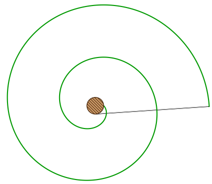
Solutions of Physics Brawl Online 2025

PhysicsBrawlOnline

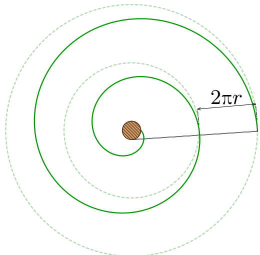
Problem 1 ... mowing

3 points

In an attempt to save effort, Verča decided to automate mowing her lawn. She firmly planted a pole in the center of the lawn and attached the lawnmower to it with a rope. The mower can move and cut only forward, but due to the rope winding around the central pole, its path will form a spiral, allowing it to cut an approximately circular area around the pole. What radius must the pole have so that the rope winds in such a way that the cut strips perfectly align with each other? The mower cuts strips of grass 0.75 m *wide; neglect the thickness of the rope.*



Verča watched DIY videos.



In the solution, it is sufficient to consider the situation along a single line connecting a given position of the mower and the center of the pole. Imagine that at a given moment, the mower passes a point and cuts a strip 0.75 m wide. After one full rotation of the pole, we want the next equally wide strip to align precisely with the previous one. For simplicity, we can assume the mower is a point tied to the rope exactly at the center of the strip. To ensure the edges meet, the rope must move exactly 0.75 m per rotation (half of that distance to reach the edge and the other half to return to the center). This means that one full rotation around the pole must shorten the rope by exactly 0.75 m, which is equal the circumference of the pole. From this equation, the radius r is then given by

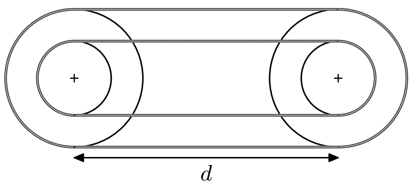
$$2\pi r = 0.75\,\mathrm{m}\,, \qquad \Rightarrow \qquad r = \frac{0.75\,\mathrm{m}}{2\pi} \doteq 0.12\,\mathrm{m} = 12\,\mathrm{cm}\,.$$

The pole must therefore have a radius $r \doteq 12\,\mathrm{cm}$.

Problem 2 ... different period

3 points

Two parallel shafts are spaced $d = 25.0\,\mathrm{cm}$ *apart. Each carries one wheel of radius* $r = 3.50\,\mathrm{cm}$ *and another of radius* $R = 6.50\,\mathrm{cm}$*. A stretched rubber band is looped around the two small wheels, and another around the two large ones. One shaft rotates at frequency* $f = 1.28\,\mathrm{Hz}$*. Find the ratio of the time for the small-wheel band to return to its initial position after one rotation to that for the large-wheel band. The rubber band does not slip on the wheels.*



Jarda was chained by his love for mechanical engineering.

For the lengths of the belts we have

$$L = 2\pi R + 2d$$

and

$$l = 2\pi r + 2d\,.$$

The belt speeds are determined by the tangential speeds at the wheel rims: for the large wheel $V = \omega R$ and for the small wheel $v = \omega r$, where $\omega = 2\pi f$ is the angular frequency of the shaft.

Thus,

$$t = \frac{l}{v} = \frac{2\pi r + 2d}{\omega r}$$

and

$$T = \frac{L}{V} = \frac{2\pi R + 2d}{\omega R}\,,$$

so the desired ratio of times is

$$\frac{t}{T} = \frac{1 + \frac{d}{\pi r}}{1 + \frac{d}{\pi R}} \doteq 1.47\,.$$

Problem 3 ... playing with gas 3 points

By what factor does the volume of an ideal gas change if its thermodynamic temperature increases by 80 % and its pressure decreases by 60 %?

Tomáš was gassed after doing so much work.

Let $\alpha = 1 + 0.8 = 1.8$ and $\beta = 1 - 0.6 = 0.4$.

We can write the equation of state for the initial and final states as

$$pV = NkT\,,$$
$$\beta pV' = Nk\alpha T\,,$$

Dividing the second equation by the first yields $V' = 4.5\,\mathrm{V}$. The final volume is therefore 4.5 times the initial volume of the gas.

Problem 4 ... fast serve 4 points

In table tennis, it is sometimes tactical to surprise your oponent with a long and fast serve. The first impact of the ball is as close as possible to the edge of the serving table, the second impact is as close as possible to the opponent's edge. What is the shortest possible ball flight time from one side of the table to another? Do not consider air resistance or ball rotation. Look up all the necessary parameters.

Jarda wants to serve a defeat as a second course to his opponent.

Consider how the ball moves across the table. It first bounces somewhere on the server's half and then on the opponent's half. Between these two bounces, it follows the trajectory of a projectile motion, which satisfies the equations for the horizontal direction x and the vertical direction y

$$x = v_0 \cos \alpha t \,,$$
$$y = v_0 \sin \alpha t - \frac{1}{2} g t^2 \,,$$

where v_0 is the initial speed and α is the angle at which the ball is launched with respect to the horizontal plane. The time after which the ball hits the table again is

$$t = \frac{2 v_0 \sin \alpha}{g} \,.$$

It is clear that the larger the vertical component $v_0 \sin \alpha$, the longer the time between bounces.

In the vertical direction, however, the situation is limited by the height of the net, which according to the official rules is $H \doteq 15.25\,\mathrm{cm} = 6\,\mathrm{inch}$. The ball must pass over it, so its kinetic energy decreases by mgH. Since the horizontal velocity remains constant, this change occurs at the expense of the vertical velocity as

$$mgH = \frac{1}{2} m v_0^2 \sin^2 \alpha \,.$$

We can substitute for the velocity to obtain the time between bounces as

$$t = 2\sqrt{\frac{2H}{g}} \,.$$

This is the shortest time the ball needs to clear the net. It is also clear that if the ball were to reach a greater height, it would take even longer. The ideal case is when the ball lands right at the server's edge and then again at the far edge of the opponent's side. If the bounce occurs somewhere within the table, the ball must first travel that distance, but it still has to reach the height H, which again takes the same time t.

Therefore, the minimal time the ball needs to cross the table and clear the net is indeed

$$t = 2\sqrt{\frac{2H}{g}} \doteq 0.35\,\mathrm{s} \,,$$

regardless of the path over the net. Of course, in reality, forces opposing the motion (e.g., air resistance) rapidly decrease the ball's velocity.

Problem 5 ... old power supply

3 points

David found an old DC power supply. He was wondering about how to use it, so he tried to connect it into a circuit with 2 resistors with the resistance of $R = 10\,\Omega$. In the first case he connected them in series, then parallel. In both cases he measured the current in the circuit. In the first case he measured the current as $I_1 = 1.2\,\mathrm{A}$ and in the second one as $I_2 = 4.0\,\mathrm{A}$. David knew that the power supply is not ideal. Calculate the internal resistance of the power supply.

David sought after some nice electrical problem.

Let's label the internal resistance of the power supply as R_i. This resistance is effectively always in series with other resistances. First case resistance is thus $2R + R_\mathrm{i}$ and second case resistance is $R/2 + R_\mathrm{i}$. Voltage of the source is in both cases U, thus we can input our equations into the Ohm's law and assemble a system of 2 linear equations with 2 unknown parameters

$$U = (2R + R_i)I_1\,,$$
$$U = (R/2 + R_i)I_2\,.$$

Left sides of the equations are equal, thus must right sides be as well. Then we just solve the linear equation for R_i

$$(R/2 + R_i)I_2 = (2R + R_i)I_1\,,$$
$$R_i(I_2 - I_1) = R(2I_1 - I_2/2)\,,$$
$$R_i = R\frac{2I_1 - I_2/2}{I_2 - I_1} = 1.4\,\Omega\,.$$

Problem 6 ... firmly connected mathematical pendulums — 3 points

Consider two massless rods of length $l = 15$ cm, each suspended by one end and able to rotate freely about their suspension points. These suspension points are at the same height, separated by a distance $d = 30$ cm. The free ends of the rods are connected by a rod of mass $m = 300$ g and length d. What is the period of small oscillations if the system is displaced within the plane containing the rods? The system is in a gravitational field with acceleration g.

Lego idealized a spring as a thin rod.

There is one key thing to realize: since the massless rods both have length l and both their upper and lower ends are at a distance d, the rods will always form a parallelogram during oscillation; that is, if we think of the line connecting the attachment points of rods as the upper side.

Thus, the middle rod will always be horizontal and if one of the massless rods is displaced by φ, then the second rod will also be displaced by φ. We can, therefore, imagine that the center of mass of the middle rod is suspended on an imaginary rod which is located exactly between the two real ones (and is also displaced by the same angle).

That way, we converted this problem into the problem of a standard mathematical pendulum of length l with a period of $T = 2\pi\sqrt{l/g} = 0.78\,\mathrm{s}$.

Problem 7 ... icy cold LED lights — 3 points

David read a study stating that LEDs in a laboratory have an electrical input power of $P_\mathrm{ele} = 30\,\mathrm{pW}$, but an optical output power of $P_\mathrm{opt} = 69\,\mathrm{pW}$, which would correspond to an efficiency greater than 100%. At first, he was surprised, but then realized that part of the power comes from thermal energy. How many diodes would be needed to produce an ice cube with a volume of $V = 10\,\mathrm{cm}^3$ in one hour, assuming we have water at a temperature of $20\,^\circ\mathrm{C}$? Assume that all optical power comes from the heat of the water and the electrical source. The optical power is independent of temperature, and heat transfer from the surroundings is negligible.

David did not realize led (Czech for ice) and LED are just homophones.

It is important to realize how the cooling power P is determined. Starting from the energy conservation principle, i.e., from the fact that

$$P + P_{\mathrm{ele}} = P_{\mathrm{opt}} \quad \Rightarrow \quad P = P_{\mathrm{opt}} - P_{\mathrm{ele}} = 39\,\mathrm{pW}\,.$$

We then determine how much energy Q the water must relinquish in order to produce $10\,\mathrm{cm}^3$ of ice:

$$Q = l_{\mathrm{T}} \cdot m + m \cdot c \cdot \Delta T\,.$$

Here l_{T} is the specific latent heat of fusion, m is the mass of the ice, c is the specific heat capacity of water, and δT is the temperature difference between the water and the freezing point. We use the density of ice ρ to obtain the mass of the ice as

$$m = V\rho \doteq 9.17 \cdot 10^{-3}\,\mathrm{kg}\,.$$

Because the ice must freeze within a time $\tau = 1\,\mathrm{h}$, the required total power P_{total} is

$$\begin{aligned} P_{\mathrm{total}} &= \frac{Q}{\tau} = \frac{l_T \cdot V\rho + V\ \ \rho \cdot c \cdot \Delta T}{\tau} \\ &= \frac{334\,\mathrm{kJ{\cdot}kg^{-1}} \cdot 9.17 \cdot 10^{-3}\,\mathrm{kg} + 9.17 \cdot 10^{-3}\,\mathrm{kg} \cdot 4184\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-}1} \cdot 20\,\mathrm{K}}{1\,\mathrm{h}} \\ &\doteq 1.06\,\mathrm{W}\,. \end{aligned}$$

Finally, since we are interested in the number of diodes, we solve for n such that

$$P_{\mathrm{total}} = nP \quad \Rightarrow \quad n = \frac{1.06\,\mathrm{W}}{39\,\mathrm{pW}} \doteq 2.7 \cdot 10^{10}\,.$$

So, it seems we will have to wait a bit longer for a refrigerator powered by LEDs.

Problem 8 ... lost gas 4 points

In the examination of the surface area of porous or powdery materials, gas adsorption on the material can be utilized. A sample with a skeletal density of $1200\,\mathrm{kg{\cdot}m^{-3}}$ *and mass* $59\,\mathrm{mg}$ *is placed into a* $15\,\mathrm{ml}$ *pipette, and the air is evacuated to create a very high vacuum. Then, a* $1.1 \cdot 10^{-3}\,\mathrm{mol}$ *of nitrogen gas is introduced After reaching equilibrium, the pressure is measured to be* $0.15\,\mathrm{bar}$*. What is the surface area of the sample, assuming that the area of one adsorbed molecule is* $0.16\,\mathrm{nm}^2$*? The whole experiment is conducted at the temperature of liquid nitrogen* $T = -196\,^{\circ}\mathrm{C}$*.*

Jarda is now looking for a way to get his gas back.

In this experiment, we measure a lower pressure than what we would expect from the ideal gas law

$$p_{\mathrm{s}} = \frac{nRT}{V_{\mathrm{p}}}\,,$$

where p_{s} is the pressure in the case where no adsorption occurs, R is the molar gas constant, $T = -196\,^{\circ}\mathrm{C} = 77\,\mathrm{K}$ is the temperature of the gas, and V_{p} is the volume that the gas can occupy inside the pipette. The discrepancy arises because the nitrogen molecules that become adsorbed do not move through space and do not collide with the container walls, and therefore do not contribute to the pressure. The measured pressure p_m corresponds to the amount of free nitrogen remaining in the gas phase n_p as

$$n_\mathrm{p} = \frac{p_\mathrm{m} V_\mathrm{p}}{RT}\,,$$

where the accessible volume can be determined from the pipette volume and the sample volume as

$$V_\mathrm{p} = V - \frac{m}{\rho}\,.$$

The amount of nitrogen adsorbed onto the sample is therefore

$$n_\mathrm{ads} = n - n_\mathrm{p} = n - \frac{p_\mathrm{m}\left(V - \frac{m}{\rho}\right)}{RT}$$

If each molecule occupies an area σ„ then the total surface area of the sample is

$$A = n_\mathrm{ads} N_\mathrm{A} \sigma \doteq 72\,\mathrm{m}^2\,,$$

where the amount of substance (in moles) has been converted to the number of particles using Avogadro's constant N_A. We see that some materials can have a surface area comparable to that of a large apartment even in very small amounts. This can be used, for example, in catalytic reactions.

Problem 9 ... brokenly charged — 4 points

A particle characterized by a charge $Q = -15.0\,\mathrm{\mu C}$ and a mass $M = 50.0\,\mathrm{\mu g}$ is flying perpendicular to magnetic field lines at a speed of $v = 100\,\mathrm{m{\cdot}s^{-1}}$. At one moment, this particle decays into two particles with opposite charges. What is the ratio of the radius of curvature of the positively charged particle's trajectory to that of the negatively charged particle? The mass of the resulting positively charged particle is $m = 20.0\,\mathrm{\mu g}$, and its charge is $q = 10.0\,\mathrm{\mu C}$. Assume that after the decay, both particles retain the velocity v, and they do not interact with each other further. *Pepa was procrastinating during the history of physics class.*

The radius of curvature of a charged particle's trajectory in a magnetic field is determined from the equality of the Lorentz and centripetal forces. For a positive particle, we have

$$qvB = m\frac{v^2}{r_1} \quad \Rightarrow \quad r_1 = \frac{mv}{qB}\,.$$

The negative particle has a mass $m_2 = M - m = 30\,\mathrm{\mu g}$ and a charge $q_2 = Q - q = -25\,\mathrm{\mu C}$, therefore, its radius of curvature is

$$r_2 = \frac{(M-m)v}{|Q-q|\,B}\,.$$

The ratio of the radii of curvature is then

$$\frac{r_1}{r_2} = \frac{\frac{mv}{qB}}{\frac{(M-m)v}{|Q-q|B}} = \frac{|Q-q|\,m}{q(M-m)} = \frac{5}{3} \doteq 1.67\,.$$

Problem 10 ... creative DnD Hero 4 points

The heroes fight a necromancer in his laboratory. One of the heroes is not exactly strong; however, he is creative and so he gets the idea to knock over a wardrobe so that it would collapse onto a nearby undead. In the brink of a moment, he starts wondering whether him having the strength to knock over the wardrobe implies that the undead would subsequently have enough strength to hold the wardrobe. Suppose the wardrobe is homogeneous and has height $a = 3.5\,\mathrm{m}$, width $b = 1.5\,\mathrm{m}$ and depth $c = 60\,\mathrm{cm}$ and the hero can topple it along its width axis. Both the hero and the undead have height $h = 180\,\mathrm{cm}$ (the height at which they will act on the wardrobe). The undead is standing $d = 1.5\,\mathrm{m}$ from the wardrobe (measured from the edge of the wardrobe base that remains on the ground). What is the ratio of the force that the undead needs to hold it over its head (provided it keeps standing in the same place) to the force the hero needs to knock it over? Neglect the non-zero speed of a falling wardrobe, i.e., consider this to be a static problem; both the undead and the hero pushed perpendicularly toward the wardrobe side. *Lego has only played DnD a few times.*

The hero will need the most strength to initially tilt the wardrobe. Then the center of mass will be closer to the axis of rotation and thus the torque that the hero must exert will be smaller (assuming, of course, that the hero still has a force arm of h). In the beginning, the arm of the gravitational force will be $c/2$. Let us denote the mass of the wardrobe as m, then the hero needs to have a force of at least

$$F_h = \frac{M}{r_h} = \frac{mg}{h}\frac{c}{2}\,.$$

The line connecting the center of mass of the wardrobe and the base, around which it was overturned, formed an angle ψ with the vertical direction before it was overturned as

$$\psi = \arctan\frac{c/2}{a/2} = \arctan\frac{c}{a}\,.$$

The angle φ between the wardrobe wall and the vertical direction after it has been overturned can be calculated using the height of the undead and its distance from the base of the wardrobe as

$$\varphi = \arctan\frac{d}{h}\,.$$

Since the wall was originally vertical, this is also the angle by which the wardrobe rotated. If we want to determine the angle formed by the line connecting the center of mass and the base with the vertical direction, we need to find out the difference of these two angles. (The difference of the angles, even though the wardrobe was knocked over in the "opposite direction" to where the center of mass was in relation to the base at the beginning.)

We also need to determine the lever arm, which we get as the horizontal distance of the center of mass from the base. Their total distance (the hypotenuse in the relevant triangle that we will use) is $\sqrt{a^2+c^2}/2$ and therefore we will use sine

$$\Delta x = \frac{1}{2}\sqrt{a^2+c^2}\sin\left(\arctan\frac{d}{h} - \arctan\frac{c}{a}\right)\,.$$

We have obtained the lever arm of the gravitational force. What lever arm will a corpse have? If it pushes perpendicular to the wall of the wardrobe, the lever arm will be given by the Pythagorean theorem $r_z = \sqrt{d^2+h^2}$. Consequently, the toppling will require a force of

$$F_z = mg\frac{\Delta x}{r_z} = mg\frac{\frac{1}{2}\sqrt{a^2+c^2}\sin\left(\arctan\frac{d}{h} - \arctan\frac{c}{a}\right)}{\sqrt{d^2+h^2}}\,.$$

Therefore the ratio we are looking for is

$$\frac{F_z}{F_h} = \frac{mg\frac{\frac{1}{2}\sqrt{a^2+c^2}\sin\left(\arctan\frac{d}{h}-\arctan\frac{c}{a}\right)}{\sqrt{d^2+h^2}}}{\frac{mgc/2}{h}} = \frac{h\sqrt{a^2+c^2}\sin\left(\arctan\frac{d}{h}-\arctan\frac{c}{a}\right)}{c\sqrt{d^2+h^2}} = 2.3\,.$$

Problem 11 ... staircases in Atomium

3 points

The Atomium in Brusel is a monumental tourist attraction that resembles an elementary cell of a spatially centred lattice of crystalline iron. Instead of atoms, there are large spheres that can be entered. The spheres are connected by passages running along all edges of the cube and also from the corner spheres into the central sphere. The cube is standing on one of its vertices so that its body diagonal points perpendicular to the ground. What is the slope of the staircase leading from the second lowest sphere to the central sphere with respect to the horizontal plane of the ground?

Jarda was on a holiday in Belgium.

According to the problem statement, the Atomium has the shape of a cube of edge length a, which is oriented with its body diagonal vertical. If we denote the vertices of the cube by ABCDEFGH so that ABCD bound one face and EFGH the opposite face, choose the plane that contains the two face diagonals AC and EG. The section of the cube by this plane is a rectangle with sides a and $\sqrt{2}a$ while also containing the center of the cube. The diagonal of this rectangle is the body diagonal of the whole cube.

If we imagine this rectangle standing on its corner with its diagonal perpendicular to the ground, it is possible to infer that it also represents the positions of the spheres in the Atomium. Denote by α the angle between the longer side and the vertical; then

$$\sin\alpha = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}, \cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}\,.$$

The height at which the lower three spheres of the Atomium lie is

$$h_\mathrm{s} = a\sin\alpha$$

and their distance from the body diagonal is

$$d_\mathrm{s} = a\cos\alpha\,.$$

The height of the middle sphere above the ground is

$$h_\mathrm{p} = \frac{\sqrt{3}}{2}a\,.$$

We can now express the tangent of the angle that the stairs make with the ground as

$$\tan\theta = \frac{h_\mathrm{p}-h_\mathrm{s}}{d_\mathrm{s}} = \frac{\sqrt{3}-2\sin\alpha}{2\cos\alpha}\,,$$

$$\tan\theta = \frac{\sqrt{3}-2\frac{1}{\sqrt{3}}}{2\frac{\sqrt{2}}{\sqrt{3}}}\,,$$

$$\tan\theta = \frac{1}{2\sqrt{2}}\,,$$

from which we obtain the desired angle

$$\theta = \arctan \frac{1}{2\sqrt{2}} \doteq 19.5^\circ \,.$$

Problem 12 ... the strength of LED strips — 4 points

Jarda installed 2.5 m *long LED strips on his kitchen counter, with a power consumption of* 11 W/m *and a voltage of* $U = 12\,\mathrm{V}$. *The voltage is supplied from a transformer via a* $l = 4.8\,\mathrm{m}$ *long twin-lead cable in which the two wires are spaced* $d = 3.0\,\mathrm{mm}$ *apart. What is the magnitude of the force between the pair of wires? If the force is attractive, submit a positive value; if repulsive, submit a negative value. Assume the thickness of the metal wires to be negligible.*

Jarda was attracted by the electron force in cables while decorating his kitchenette.

As a result of current flowing through the twin cable, a magnetic field is produced around both conductors. The current in a conductor is carried by moving electrons, which experience the Lorentz force in the presence of a magnetic field. The magnitude of this force can be determined from Ampère's law for the force between two conductors

$$|F| = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d} l \,, \tag{1}$$

where $I_1 = I_2 = I$ are the currents in the individual conductors, $d = 3\,\mathrm{mm}$ is their separation, and $l = 4.8\,\mathrm{m}$ is their length. The twin cable consists of the supply and the "ground", which together form a closed circuit. The currents in the conductors therefore flow in opposite directions, making the force repulsive (left-hand rule). It remains only to determine the current in the circuit. From the statement we have a linear power consumption of the LED strips of $11\,\mathrm{W{\cdot}m^{-1}}$ and a length of 2.5 m, hence the circuit input power is $P = (11\,\mathrm{W{\cdot}m^{-1}}) \cdot (2.5\,\mathrm{m}) = 27.5\,\mathrm{W}$. From the total input power

$$I = \frac{P}{U} \,.$$

Substituting into (1) gives

$$|F| = \frac{\mu_0}{2\pi} \frac{P^2}{U^2 d} l \doteq 1.7\,\mathrm{mN} \,.$$

Since the force between the conductors is repulsive, it is taken as negative in the statement: $F = -1.7\,\mathrm{mN}$.

Problem 13 ... skipping pizza — 4 points

On a spring of stiffness $k = 100\,\mathrm{N{\cdot}m^{-1}}$ *with a platform there is a pizza of mass* $m = 0.40\,\mathrm{kg}$. *The spring with the platform is, by a built-in mechanism, compressed to an initial displacement* $x_0 = 0.020\,\mathrm{m}$ *from the equilibrium position. The mechanism then releases the spring, which launches the pizza and immediately after the launch compresses the spring back to the initial displacement* x_0. *The pizza then lands back on the platform, compresses the spring and is launched again. Immediately after each launch the mechanism pulls the spring to the maximal* *displacement from the equilibrium position that was reached after the pizza's impact. This process repeats.*

To what displacement will the spring be compressed by the mechanism after the launch at which the pizza first exceeds the height $h = 1.0\,\mathrm{m}$ above the equilibrium position? Neglect air resistance. Assume that all kinetic energy of the pizza and the potential energy of the spring are converted into each other ideally (without losses) and that the mass of the platform and the spring is negligible compared to the mass of the pizza.

David was thinking about making the preparation of Hawaiian pizza more efficient.

In order for the pizza to clear the height $h = 1\,\mathrm{m}$, it must hold that the energy delivered by the spring E_k is greater than or equal to the potential energy of the pizza at height h, that is, $E_\mathrm{k} \geq E_\mathrm{p}$.

For the potential energy we have

$$E_\mathrm{p} = mgh\,.$$

This gives the potential energy at height $h = 1\,\mathrm{m}$ as $E_\mathrm{p} \doteq 3.92\,\mathrm{J}$.

The energy of the spring can be calculated as

$$E_\mathrm{k} = \frac{1}{2}kx^2\,.$$

For the first launch this yields $E_\mathrm{k} = 0.02\,\mathrm{J}$. Writing the first four launches

$$\begin{aligned} E_{\mathrm{k}_1} &= 0.02\,\mathrm{J}\,,\\ E_{\mathrm{k}_2} &= 0.04\,\mathrm{J}\,,\\ E_{\mathrm{k}_3} &= 0.08\,\mathrm{J}\,,\\ E_{\mathrm{k}_4} &= 0.16\,\mathrm{J}\,. \end{aligned}$$

We see that the energy grows according to the recurrence relation

$$E_{\mathrm{k}_n} = 2E_{k_{n-1}}\,.$$

Writing the n-th term of this recurrence and comparing to the potential energy gives

$$2^{n-1}0.02 \geq 3.92\,.$$

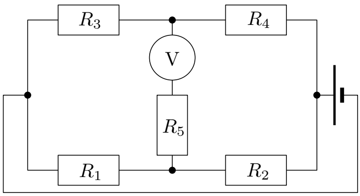
Thus $n \geq 8.62$. Hence after the 9th launch the pizza will exceed the height h. For the 9th term $E_{\mathrm{k}_9} = 5.12\,\mathrm{J}$. To compute the spring compression x_9 we solve the spring-energy equation for x

$$x_9 = \sqrt{\frac{2E_{\mathrm{k}_9}}{k}}\,.$$

This yields that the pizza will be launched above $h = 1\,\mathrm{m}$ when the spring is compressed to $x_9 = 0.32\,\mathrm{m}$.

Problem 14 ... voltage on the voltmeter 4 points

What voltage will the voltmeter show in the circuit shown in the diagram? The source voltage is $U_0 = 4.50\,\mathrm{V}$, the source has negligible internal resistance, the voltmeter is ideal, and the resistors have the following resistance values: $R_1 = 1.00\,\Omega$, $R_2 = 2.00\,\Omega$, $R_3 = 4.00\,\Omega$, $R_4 = 2.00\,\Omega$ and $R_5 = 5.00\,\Omega$. Provide the result as a positive number.



Karel was thinking about using the voltmeter incorrectly.

An ideal voltmeter has infinite internal resistance. We may therefore neglect resistor R_5 and treat the branch containing the voltmeter as open.

The voltmeter reading is given by the potential difference between the nodes. Denote the potential at the node between resistors R_1 and R_2 by φ_{12}, and analogously φ_{34} between R_3 and R_4. Taking the potential relative to the positive terminal, we obtain

$$\varphi_{12} = \frac{R_1}{R_1 + R_2} U_0 = 1.50\,\mathrm{V}\,,$$

$$\varphi_{34} = \frac{R_3}{R_3 + R_4} U_0 = 3.00\,\mathrm{V}\,.$$

Thus the absolute value of the voltage on the voltmeter is given by the difference of the potentials

$$U_\mathrm{V} = |\varphi_{12} - \varphi_{34}| = 1.50\,\mathrm{V}\,.$$

The voltmeter will read $1.50\,\mathrm{V}$. The problem states the result as positive — in practice the sign would depend on the polarity of the voltmeter connection.

Problem 15 ... radar operator problems 4 points

A radar operator transmitted a pulse at frequency $f = 50\,\mathrm{MHz}$ toward an enemy aircraft flying at an altitude $h = 20\,000\,\mathrm{ft}$ above the ground. After a time $t = 0.10\,\mathrm{ms}$ the pulse returned with a frequency shifted by $\Delta f = 190\,\mathrm{Hz}$ relative to the transmitted pulse. Determine how long it will take until the enemy aircraft is directly above the radar operator, assuming the aircraft flies uniformly in a straight line and parallel to the ground toward the point directly above the radar.

Dodo was watching the (weather) radar.

First, consider the situation described in the problem statement. In the vertical direction, an airplane is approaching the zenith at altitude h with speed v, and at the point where the pulse is reflected from the airplane, the zenith–satellite–airplane angle is φ (fig. 1). When the pulse is reflected, a double frequency shift occurs—the airplane first receives the pulse as an observer moving toward the source with speed $v_{\parallel}$, and then (upon reflection) it "emits" it as a source moving directly toward the observer (the satellite) with speed $v_{\parallel}$. The difference between the emitted and received frequencies is $\Delta f = f' - f$. Since $\Delta f \ll f$, it is sufficient to consider the non-relativistic Doppler effect.

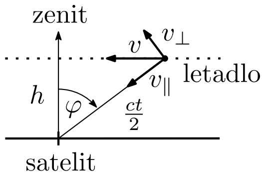


Figure 1: Sketch of the situation from the problem

From figure 1 it is clear that the airplane is not moving directly toward the observer. Its velocity can be decomposed into a radial velocity $v_{\parallel} = v \sin\varphi$, whose direction is parallel to the satellite–airplane line, and a tangential velocity $v_{\perp}$, whose direction is perpendicular to the radial one. If we represent the electromagnetic wave (pulse) by means of its wavefronts, we find that the tangential component of the velocity does not affect how the observer perceives the transmitted signal. Therefore, only the radial component $v_{\parallel}$ is what we are after.

When an observer approaches the source directly with speed $v_{\parallel}$, the observer perceives the wave with the same physical wavelength but with a different speed, and therefore with a different frequency. Thus,

$$\frac{f_1}{c + v_{\parallel}} = \frac{f}{c} \to f_1 = \frac{c + v_{\parallel}}{c} f\,.$$

When the source approaches the observer, the physical wavelength shortens, but the wave speed c remains constant. During one period T, the original wavelength λ_1 shortens to

$$\begin{aligned} \lambda_2 &= \lambda_1 - v_{\parallel} T = \lambda_1 - v_{\parallel}\frac{\lambda_1}{c} = \lambda_1 \frac{c - v_{\parallel}}{c}\,,\\ f_2 &= \frac{c}{c - v_{\parallel}} f_1\,. \end{aligned}$$

Combining both Doppler shifts, we obtain

$$f' = f_2 = \frac{c}{c - v_{\parallel}} \frac{c + v_{\parallel}}{c} f = \frac{c + v_{\parallel}}{c - v_{\parallel}} f.$$

From this, we can determine

$$\begin{aligned} v = \frac{v_{\parallel}}{\sin\varphi} &= \frac{f' - f}{f' + f}\frac{c}{\sin\varphi} = \\ &= \frac{\Delta f}{2f + \Delta f}\frac{c}{\sin\varphi} \approx \frac{\Delta f}{2f}\frac{c}{\sin\varphi}\,, \end{aligned}$$

where we used the fact that $\Delta f \ll f$. While the pulse reaches the airplane during the time $t/2$, it covers the distance

$$\frac{h}{\cos\varphi} = \frac{ct}{2} \to \cos\varphi = \frac{2h}{ct}\,.$$

After the pulse is reflected from the airplane, a time $t/2$ again elapses before the pulse returns to the satellite. The time t_2 from receiving the pulse until the airplane reaches the zenith is then

$$\frac{ct}{2}\sin\varphi = v\left(t_2 + \frac{t}{2}\right)\,,$$
$$t_2 = \left(\frac{c}{v\sin\varphi} - 1\right)\frac{t}{2}.$$

After rearranging,

$$t_2 = \left(\frac{2f}{\Delta f}\sin^2\varphi - 1\right)\frac{t}{2} = \left(\frac{2f}{\Delta f}(1-\cos^2\varphi) - 1\right)\frac{t}{2} =$$
$$= \left(\frac{2f}{\Delta f}\left(1 - \frac{4h^2}{c^2t^2}\right) - 1\right)\frac{t}{2} \doteq 22\ \mathrm{s}\,.$$

Problem 16 ... spinning electrons 4 points

Danka was working in the lab with flask filled with low-pressure argon, into which a collimated electron beam was introduced. The beam ionized the argon, and during recombination, the path of the electron beam could be observed due to luminescence. Danka adjusted the beam so that its trajectory took the shape of a helix with a radius of $5.00\,\mathrm{cm}$ *and a constant pitch of* $5.00\,\mathrm{cm}$. *The accelerating voltage of the electrons was* $200\,\mathrm{V}$. *What was the magnitude of the magnetic induction (magnetic field) in the flask produced by the external magnets under these conditions? Assume that the magnetic induction is constant throughout the entire volume of the flask and has a uniform direction.*

Danka did an experiment for he Practical Course class, but it was a long time ago.

If electrons are accelerated through a potential U, their total speed follows from energy conservation:

$$Ue = \frac{1}{2}m_\mathrm{e}v^2\,,$$
$$v = \sqrt{\frac{2Ue}{m_\mathrm{e}}}\,.$$

This velocity can be decomposed into a component parallel to the magnetic induction (v_x) and a component perpendicular to it (v_y). The given pitch is the distance s travelled in the x-direction while the electron describes a circle in the plane perpendicular to the magnetic induction B. Hence

$$v^2 = v_x^2 + v_y^2\,,$$
$$\frac{s}{v_x} = \frac{2\pi R}{v_y}\,,$$

where R is the radius of the circle described in the plane perpendicular to B. From this we obtain v_y as

$$v_y = \frac{v}{\sqrt{1+\left(\frac{s}{2\pi R}\right)^2}}\,.$$

We also have that electrons are deflected in the magnetic field according to

$$Bev_y = \frac{m_\mathrm{e}\left(v_y\right)^2}{R}\,.$$

Combining the above equations yields

$$B = \frac{m_\mathrm{e} v_y}{eR} = \frac{m_\mathrm{e}\frac{v}{\sqrt{1+\left(\frac{s}{2\pi R}\right)^2}}}{eR} = \frac{\sqrt{2Uem_\mathrm{e}}}{eR\sqrt{1+\left(\frac{s}{2\pi R}\right)^2}} \doteq 0.942\,\mathrm{mT}\,.$$

Problem 17 ... overturned outhouse

4 points

It was windy and Kuba was walking along a footpath. Suddenly he saw an overturned outhouse, and immediately began to wonder what wind speeds are required to tip it over. Approximate the outhouse as a homogeneous rectangular block of height $a = 244\,\mathrm{cm}$, with a square base of side length $b = 152\,\mathrm{cm}$ and mass $m = 140\,\mathrm{kg}$; the drag coefficient has the value $C = 1.05$. Assume that the wind blows perpendicularly onto one of the side walls and that the axis of rotation is fixed at the edge of the outhouse touching the ground opposite the side on which the wind is blowing. Take the air density to be $\rho = 1.225\,\mathrm{kg{\cdot}m^{-3}}$. What is the minimum wind speed required for the outhouse to start tipping?

Jarda P. went to the relieve himself.

First, we determine the force exerted by the air impacting the outhouse. This force is given by the well-known formula

$$F_v = \frac{1}{2} C\rho S v^2\,,$$

where S is the effective area, i.e., the area the air stream hits. Next, the outhouse has mass m. Therefore, we have the gravitational force of magnitude

$$F_g = mg\,.$$

The elements of the drag force act along the wall of the outhouse; we may replace them by a single equivalent force acting at the center of the wall. The vector of this force can be shifted along its line of action into the center of mass, where gravity acts as well, without changing its dynamical effect. The aerodynamic drag acts horizontally along the ground on which the outhouse stands, while the gravitational force acts vertically perpendicular to it. Adding these two forces gives the resultant.

Since the outhouse is approximated as a homogeneous rectangular box, we know that its center of mass lies in its geometric center. The resultant, which acts at the center of mass, can be joined with the axis of rotation by a segment whose length is half of the diagonal. By “diagonal” we do not mean the body diagonal of the box, but the diagonal of the rectangle orthogonal to the rotation axis.

The rotation of the outhouse is influenced by the component perpendicular to this connecting segment, i.e., the diagonal. If this component points toward the ground, the outhouse is being pressed down; if it points in the opposite direction, the outhouse will begin to tip over. The critical point—when tipping becomes just barely possible—occurs when the perpendicular component of the resultant is zero, i.e., when the resultant is parallel to the diagonal.

Let the angle between the gravitational force and the resultant be φ. Then we have

$$\tan(\varphi) = \frac{F_\mathrm{v}}{F_g} = \frac{\frac{1}{2}C\rho S v^2}{mg} \,. \tag{2}$$

The diagonal forms an angle θ with the longer edge of the outhouse. Thus,

$$\tan(\theta) = \frac{b}{a} \,. \tag{3}$$

Because we are looking for the critical point, we have $\varphi = \theta$. Now we compare relations (2) and (3). We also substitute $S = ab$. Thus we get

$$\frac{C\rho a b v^2}{2mg} = \frac{b}{a} \,.$$

By rearranging, we obtain

$$v = \sqrt{\frac{2mg}{C\rho a^2}} \doteq 18.9\,\mathrm{m{\cdot}s^{-1}} \,.$$

Problem 18 ... potassium cube 4 points

Potassium $^{40}\mathrm{K}$ *has a half-life of* $T = 1.3 \cdot 10^9$ years. *It transforms via* β^- *decay into stable calcium isotope* $^{40}\mathrm{Ca}$ *with a probability of* 89 %, *and via* β^+ *decay into stable argon isotope* $^{40}\mathrm{Ar}$ *with a probability of* 11 %. *How much* $^{40}\mathrm{K}$ *would be required in a cube to produce enough argon that would fill a room of dimensions* $a = 5.0\,\mathrm{m}$, $b = 10\,\mathrm{m}$ *and* $c = 3.0\,\mathrm{m}$, *assuming an argon particle of* $\rho = 1.0 \cdot 10^{25}\,\mathrm{m^{-3}}$, *in one year? Assume that argon is released from the entire volume. The molar mass of* $^{40}\mathrm{K}$ *is* $M = 40\,\mathrm{g{\cdot}mol^{-1}}$.

David wondered how many bananas could he eat before choking on them.

To start with, we calculate the number of particles N contained in the required mass of potassium m:

$$N = N_A \frac{m}{M} \,,$$

where N_A is Avogadro's constant. Now we use the law of radioactive decay to obtain the number of particles N_r that will decay during the time t:

$$N_r = N\left(1 - e^{-\frac{\ln 2}{T}t}\right) \,.$$

The resulting particles include both calcium and argon, so we must multiply the total number of transformed particles by the probability p of producing argon. This gives the number of argon atoms N_{Ar} released through radioactive decay of potassium during time t. For the required density, this number must be equal to

$$N_{\mathrm{Ar}} = \rho abc\,.$$

By combining the equations above, we obtain

$$\rho abc = pN_A \frac{m}{M}\left(1 - e^{-\frac{\ln 2}{T}t}\right)\,,$$

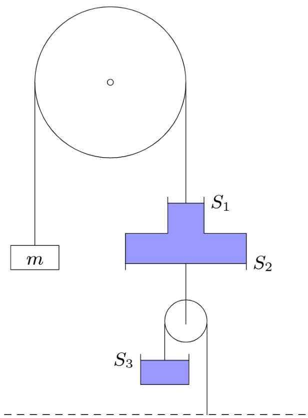
from which we solve for m:

$$m = \frac{\rho abcM}{pN_A\left(1 - e^{-\frac{\ln 2}{T}t}\right)} = 1.7 \cdot 10^{12}\,\mathrm{kg}\,.$$

Problem 19 ... thermodynamical pulleys

4 points

We have a steady configuration as shown in the figure. Both gas vessels are rigidly fixed and each contains $n = 1\,\mathrm{mol}$ *of gas at a temperature* $T = 20\,\mathrm{^\circ C}$*, which is also the ambient temperature where the atmospheric pressure applies. What is the volume of the gas in the lower piston? Neglect the weight of everything except the weight* $m = 2.5\,\mathrm{kg}$*. The areas in the figure are* $S_1 = 15\,\mathrm{cm^2}, S_2 = 50\,\mathrm{cm^2}, S_3 = 25\,\mathrm{cm^2}$.



Lego tried to mix-and-match.

First, we look at the upper vessel. Since we neglect gravity, we also neglect the hydrostatic pressure of the gas in it, so the pressure at the upper piston is equal to the pressure at the lower piston. From the force equilibrium on the first rope, it follows that this pressure must be

$$\Delta p_1 = \frac{mg}{S_1}\,,$$

smaller than the atmospheric pressure. The same difference will appear at the lower piston as well, so the force pulling the second piston upward is $F_2 = mgS_2/S_1$.

This force pulls upward on the free pulley, which splits this force between the two sides of the rope, so the piston of the lower vessel is pulled upward by the rope with a force $F_2/2$. Of course, the piston is also acted on by atmospheric pressure, so the pressure in the piston is

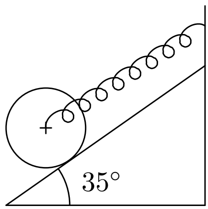
$$p = p_\mathrm{a} - \frac{F_2}{2S_3} = p_\mathrm{a} - \frac{mgS_2}{2S_1S_3}\,.$$

It remains to substitute into the equation of state:

$$V = \frac{nRT}{p} = \frac{nRT}{p_\mathrm{a} - \frac{mgS_2}{2S_1S_3}} \doteq 0.029\,\mathrm{m}^3 = 29\,\mathrm{l}\,.$$

Problem 20 ... wired wheel 5 points

A hoop with a mass of $1.3\,\mathrm{kg}$ *and a diameter of* $73\,\mathrm{cm}$ *is placed on an inclined plane, tilted at an angle of* $35\,^\circ$ *to the horizontal. The hoop has an axis of symmetry with a shaft that allows it to rotate freely. The shaft is connected to a spring with a spring constant of* $2.5\,\mathrm{N{\cdot}m^{-1}}$, *which runs parallel to the plane and points upwards. If the hoop is displaced downwards from its equilibrium position, what will be its oscillation period? Assume that there is no slipping between the hoop and the plane.*



Jarda found a truly unique item in a souvenir shop.

By the condition that the wheel does not slip, we mean that the tangential velocity of the points on the rim of the wheel relative to its center is the same as the velocity of the wheel's center of mass v relative to the ground

$$v = \omega R\,,$$

where ω is the angular velocity of the wheel and R its radius. If this condition is satisfied, the wheel rolls on the surface without slipping.

The kinetic energy of such motion of the wheel consists of translational and rotational components

$$E_\mathrm{k} = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}mR^2\omega^2 = \frac{1}{2}(2m)v^2\,,$$

where we substituted the moment of inertia of a hoop

$$J = mR^2\,.$$

We can see that when rolling, the hoop behaves like a point mass but with an effective doubled mass, because part of the energy goes into rotational motion.

The period of oscillations of a point mass m on a spring with stiffness k does not depend on the direction relative to the gravitational field. Gravity only determines how much the spring is stretched compared to its rest length. But since this force is constant and independent of position and time, it has no effect on the oscillation frequency. The period of oscillation of a point mass is therefore

$$T_\mathrm{b} = 2\pi\sqrt{\frac{m}{k}}\,.$$

Our hoop, when rolling, has an effective mass of $2m$, which is enough to substitute into this equation, giving the final result of the problem

$$T = 2\pi\sqrt{\frac{2m}{k}} \doteq 6.4\,\mathrm{s}\,.$$

Problem 21 ... particles

4 points

We have a relativistic proton for which the detector measured an energy of $E = 5.12\cdot 10^{-10}\,\mathrm{J}$. What must be the magnitude of the magnetic flux density B so that the particle's trajectory has a radius of curvature $R = 1.11\,\mathrm{m}$?

Vítek heard he could possibly avoid taking the Practical Course IV class.

The energy of a relativistic particle can be expressed as

$$E = mc^2 = \gamma m_0 c^2\,,$$

where γ is the Lorentz factor and m_0 is the invariant mass.

The magnetic field bends the trajectory of the proton, so we write the equality of the magnetic force and the centrifugal force as

$$\begin{aligned} F_m &= F_c m_0\gamma\frac{v_\perp^2}{R} \\ &= qBv_\perp\,, \\ B &= \frac{m_0\gamma v_\perp}{qR}\,. \end{aligned}$$

The perpendicular velocity $v_\perp$ is obtained from the relation

$$\begin{aligned} E = m_0\gamma c^2 &= \frac{m_0c^2}{\left(1-\frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}}} \\ v_\perp &= c\sqrt{1-\frac{m_0^2c^4}{E^2}}\,. \end{aligned}$$

Substituting this expression for the velocity into the expression for the magnetic-field induction yields

$$B = \frac{E}{qRc}\left(1-\frac{m_0^2c^4}{E^2}\right)^{\frac{1}{2}}\,.$$

After substituting the numerical values we obtain $B \doteq 9.18\,\mathrm{T}$.

Problem 22 ... modified solar system 5 points

Suppose that the Sun had a temperature of $T_2 = 8\,000\,\mathrm{K}$. By what percentage would the orbital period of Jupiter need to increase in order for it to receive the same power output as it currently does? Assume that Jupiter's orbit around the Sun is circular and that all other properties of the Sun remain unchanged.

Danka has tried to come up with interesting problems.

We can determine the power output of the Sun L using the Stefan–Boltzmann law

$$L = 4\pi R_\mathrm{S}^2 \sigma T_\mathrm{S}^4\,,$$

where R_S denotes the radius of the Sun, σ is the Stefan–Boltzmann constant, and T_S is the temperature of the Sun. From this luminosity, the power incident on the surface of Jupiter is

$$P_\mathrm{J} = \frac{L}{4\pi R_\mathrm{o}^2} \pi R_\mathrm{J}^2\,,$$

where R_o is the distance between the Sun and Jupiter, and R_J is the radius of Jupiter. From the fact that the incoming power must be equal at the original and the new temperature, we can determine the new distance R_N of Jupiter from the Sun as

$$\frac{4\pi R_\mathrm{S}^2 \sigma T_2^4}{4\pi R_\mathrm{N}^2} \pi R_\mathrm{J}^2 = \frac{4\pi R_\mathrm{S}^2 \sigma T_1^4}{4\pi R_0^2} \pi R_\mathrm{J}^2$$

$$R_\mathrm{N} = \frac{T_2^2}{T_1^2} R_0\,,$$

where T_1 is the original temperature of the Sun and R_0 is Jupiter's original distance from the Sun. The original temperature of the Sun could be looked up (it is not itself listed in the table of constants), but the table does contain the mean radiative power of the Sun $L_\odot$, from which we can compute T_S using the expression above as

$$T_1 = \left(\frac{L_\odot}{4\pi R_\mathrm{S}^2 \sigma}\right)^{\frac{1}{4}} = 5\ 772\ \mathrm{K}\,.$$

Next, we use the equality of gravitational and centrifugal force, which holds because in this scenario, Jupiter's orbit around the Sun circular

$$G \frac{M_\mathrm{S} M_\mathrm{J}}{R_o^2} = \frac{4\pi^2}{t^2} R_o\,,$$

where M_S is the mass of the Sun, M_J the mass of Jupiter, and t the orbital period of Jupiter, which we can express from this equation as

$$t = \sqrt{\frac{4\pi^2 R_o^3}{G M_\mathrm{S} M_\mathrm{J}}}\,.$$

Now we just need to determine the ratio of the original and new periods:

$$\frac{t_\mathrm{N}}{t_0} = \sqrt{\frac{R_\mathrm{N}^3}{R_0^3}} = \frac{T_2^3}{T_1^3} = 2{,}66\,.$$

Thus, the orbital period of Jupiter would have to increase by 166 percent. The first equality in the equation above is actually Kepler's third law, which we have derived through this procedure for a circular trajectory (although its validity is more general); we could have simply substituted into it.

Problem 23 ... harmonical friction 5 points

As is well known, the irregularities of contacting surfaces cause friction. A very simple one-dimensional model of such a surface can be represented by the graph of a harmonic function with a period of $d = 20.0\,\mu\mathrm{m}$ and an amplitude of $h = 1.00\,\mu\mathrm{m}$. If we place two identical "surfaces" of this type against each other, they interlock so that they follow the same curve. Determine the coefficient of static friction of such a system—that is, the ratio F/F_n between the horizontal force F acting on the upper body (at which the system begins to move continuously) and the normal force F_n pressing the upper body downward. The lower body is fixed and does not move. The model surfaces are perfectly smooth.

Kuba tried to formulate a simple friction model.

Firstly, let us consider a similar but slightly easier problem. Instead of a sinusoid, imagine sawtooth surfaces that make an angle α with the horizontal axis (Fig. 2).

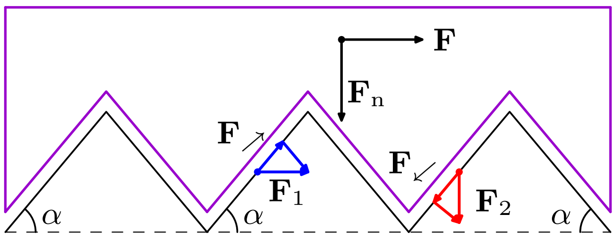


Figure 2: Two sawtooth surfaces.

If a horizontal force F acts on the sawteeth and the body does not move, the resultant of the force F and the force exerted by the sawteeth must be zero. From the symmetry of the sawteeth, it follows that each sawtooth exerts a horizontal force

$$F_1 = \frac{F}{N},$$

where N is the number of sawteeth considered. When the body starts moving, in the limiting case it will move along the direction of the sawtooth (Fig. 2). The component of the force F_1 in this direction is

$$F_{\nearrow} = F_1 \cos\alpha = \frac{F}{N}\cos\alpha\,.$$

Opposing this force is the normal force pressing the body downward. If a normal force F_n acts on the body, each sawtooth experiences a force $F_2 = \frac{F_n}{N}$. The component of the force F_2 in the direction down the sawtooth is

$$F_{\swarrow} = F_2 \sin\alpha = \frac{F_n}{N}\sin\alpha\,.$$

We consider the surfaces between the sawteeth to be perfectly smooth, so in the limiting case motion begins when

$$F_{\nearrow} = F_{\swarrow}\,,$$
$$\frac{F}{N}\cos\alpha = \frac{F_n}{N}\sin\alpha\,,$$

from which we obtain the coefficient of friction

$$\mu = \frac{F}{F_n} = \tan\alpha\,.$$

Now we return to the original problem. The interaction between the two surfaces occurs at their point of contact and is mediated through the plane that is perpendicular to both surfaces and passes through the contact point. Less formally, when working with the forces that two bodies exert on each other, we draw the tangent at their contact point and then consider forces across it. This is exactly what we have solved so far—the sawteeth from the first part of the solution are in fact the tangents at the contact point. The smallest horizontal force required to start the body moving must be sufficient to overcome the "friction of each sawtooth." Since the coefficient of friction increases with the angle α, we need to find the "sawtooth" on the given sinusoid with the largest angle α; in other words, we seek the tangent with the greatest slope.

Consider the sinusoid from the problem statement, which satisfies

$$y = h\sin\left(2\pi\frac{x}{d}\right)\,, \tag{4}$$

where y is the vertical displacement, x is the horizontal coordinate, and d is the wavelength given in the problem. To determine the slope of the tangent for different values of x, we differentiate equation (4) with respect to x

$$\tan\left(\alpha(x)\right) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\pi h}{d}\cos\left(2\pi\frac{x}{d}\right)\,.$$

The range of the cosine function is the interval $\langle -1, 1\rangle$, so the maximum value of friction occurs when $\cos\left(2\pi x/d\right) = 1$, which happens, for example, at $x = 0$. The coefficient of friction for the surfaces in question is then

$$\mu = \frac{2\pi h}{d} \doteq 0.314\,.$$

Problem 24 . . . tense oscillations — 5 points

Terka was playing with a plate capacitor consisting of two parallel metallic plates separated by a distance $d = 10.0\,\mathrm{cm}$ and held at a potential difference $U = 200\,\mathrm{V}$, when suddenly her earring fell into the capacitor. She looked in tensely and observed an interesting phenomenon—the earring began to jump back and forth between the plates.

Determine the steady-state frequency of these oscillations. Each plate of the capacitor has an area $S = 1.00\,\mathrm{m^2}$. Model the earring as a conducting sphere of radius $a = 2.00\,\mathrm{mm}$ and mass $m = 0.15\,\mathrm{g}$. Collisions of the sphere with the plates are characterized by a coefficient *of restitution $\beta = 0.98$. Upon contact with a plate, the sphere acquires a charge $q = \pm 4.45 \cdot 10^{-11}\,\mathrm{C}$. Neglect the influence of gravity on the motion.*

David misses doing experiments in a lab.

The energy acquired by the sphere during its travel from one plate to the other is

$$\Delta E_{\mathrm{kin}} = U\ q = \frac{1}{2} m v_0^2 \quad \Rightarrow \quad v_0 = \sqrt{\frac{2Uq}{m}}\,.$$

From this relation, we determine v_0 as the speed corresponding to the energy increment. This idea will assist in the subsequent calculations. When the sphere collides with the opposite plate, a fraction of its kinetic energy is dissipated as heat. Therefore, in the steady state, one has

$$v_2 = \beta \sqrt{\frac{1/2 m v_2^2 + \Delta E_{kin}}{1/2 m}} = \beta \sqrt{v_2^2 + v_0^2}\,,$$

where v_2 denotes the speed immediately after the rebound. Further manipulation gives

$$\frac{v_2}{v_0} = \frac{\beta}{\sqrt{1-\beta^2}}\,.$$

First, we determine the time τ required for the sphere to travel from one plate to the other, i.e., to cover the distance $d - 2a$ as

$$\frac{d-2a}{\tau} = \overline{v} = \frac{v_2 + v_2/\beta}{2} \quad \Rightarrow \quad \tau = 2 \frac{d-2a}{v_2 + v_2/\beta}\,.$$

From this, we obtain the oscillation frequency f as

$$f = \frac{1}{2\tau} = \frac{v_2 + v_2/\beta}{4(d-2a)} = \frac{v_0}{4(d-2a)} \sqrt{\frac{1+\beta}{1-\beta}} = \frac{\sqrt{2Uq/m}}{4(d-2a)} \sqrt{\frac{1+\beta}{1-\beta}} \doteq 0.282\,\mathrm{Hz}\,.$$

Problem 25 ... bizzare yo-yo adventure — 5 points

A yo-yo is made of two cylinders with radius R connected by a small axle with radius r. The ratio between R and r is $q = 3.00$. All three cylinders have the same width; they are homogeneous, made of the same material. A massless, infinitely thin string of length $L = 1.25\,\mathrm{m}$ is wound around the small axle, with one end of the string firmly attached to it. When the yo-yo is released and allowed to unwind freely, after what time τ will it return to its original position? Assume that at the moment of release, the tangent to the free end of the string is vertical. Also assume that $r \ll L$, so that the effects at the turning point of the yo-yo's motion can be neglected.

Petr was brainstroming competition problems.

First a brief comment on the motion under consideration. In the first phase the yo-yo descends to the end of the string so that the string remains tangent to the yo-yo. Then the yo-yo can be imagined to swing like a pendulum about the lower end of the string through an angle of π to the other side of the string. In the third phase it then climbs back up on the other side. Since the first and third phases take equal time and the second phase is neglected, we may consider only the time $\tau/2$ required for the descent; the total time τ is twice that.

The forces acting on the yo-yo are gravity $F = mg$ and the tension $-T$ from the string (downward is chosen positive). Hence the translational equation of motion is

$$mg - T = ma\,,$$

where a is the downward acceleration of the yo-yo. There is also a nonzero net torque, and by the rotational form of Newton's second law

$$M = \varepsilon J\,,$$

where ε is the angular acceleration and J is the moment of inertia. Choosing the yo-yo axis as the reference point for moments gives $M = Tr$. Denoting by φ the angular displacement from the start, we have $\varepsilon = \ddot{\varphi}$ and therefore the differential equation

$$\ddot{\varphi} = \frac{Tr}{J} \implies \varphi = \frac{Tr}{2J}t^2 + c_1 t + c_2\,.$$

With initial conditions of zero angular displacement and zero angular velocity the constants c_1, c_2 vanish, then

$$\varphi = \frac{Tr}{2J}t^2\,.$$

Now consider the maximum angular displacement $\varphi_{\max}$ reached when the yo-yo has paid out the full string length L. A simple geometric argument shows that this is 2π times the ratio of the string length L to the circumference of the smaller drum, hence

$$\varphi_{\max} = 2\pi \frac{L}{2\pi r} = \frac{L}{r}\,.$$

This angular displacement $\varphi_{\max}$ is reached at the end of the descent, i.e. at time $\tau/2$, giving

$$\frac{L}{r} = \frac{Tr}{8J}\tau^2\,.$$

From this we solve for the unknown tension T

$$T = \frac{8JL}{r^2\tau^2}\,,$$

and substitute into the equation of motion:

$$mg - \frac{8JL}{r^2\tau^2} = ma\,.$$

Since the yo-yo undergoes uniformly accelerated motion, the displacement during the descent satisfies

$$\frac{1}{2}a\frac{\tau^2}{4} = L\,.$$

Solving for a and substituting into the equation above eliminates the remaining unknowns, yielding

$$mg - \frac{8JL}{r^2\tau^2} = \frac{8mL}{\tau^2}\,,$$

which rearranges to

$$\tau = 2\sqrt{\frac{2L}{g}}\sqrt{1 + \frac{J}{mr^2}}\,.$$

The final step is to express the moment of inertia J. For a solid cylinder of mass m' and radius r' the moment of inertia is given as

$$j = \frac{1}{2}m'r'^2\,.$$

Our yo-yo is composed of three glued cylinders. Since they rotate about the same axis the total moment of inertia J is the sum of the moments of the small drum J_r and the two large cylinders J_R. Denoting the small drum mass by μ and the mass of one large cylinder by M,

$$J = MR^2 + \frac{1}{2}\mu r^2\,.$$

Assuming homogeneous cylinders of equal thickness, the mass ratio equals the ratio of the base areas, so

$$\frac{M}{\mu} = \frac{R^2}{r^2} = q^2\,.$$

Also $2M + \mu = m$. Using these relations we obtain

$$M = \frac{q^2 m}{2q^2 + 1}\,,$$
$$\mu = \frac{m}{2q^2 + 1}\,.$$

Substituting into J yields

$$J = \frac{1}{2}\left(\frac{2q^4 + 1}{2q^2 + 1}\right)mr^2\,.$$

Inserting this into the expression for τ gives the final result

$$\tau = 2\sqrt{\frac{2L}{g}}\sqrt{1 + \frac{1}{2}\frac{2q^4 + 1}{2q^2 + 1}} \doteq 2.32\,\mathrm{s}\,.$$

Comment on neglecting the yo-yo's swing time

In the solution we neglected the time t' needed for the yo-yo to swing as a pendulum about the lower end of the string before it climbs back up. We now justify that approximation.

An exact analytical treatment is infeasible because the motion is a physical pendulum with large amplitude (specifically amplitude $\pi/2$). We therefore estimate the swing time. Let ω be the angular speed at the end of the descent and Ω the mean angular speed during the swing. Clearly

$$\Omega > \omega \implies t' = \frac{\pi}{\Omega} < \frac{\pi}{\omega}\,.$$

With $\omega = v/r$, $v = a\tau/2$, and $a = 8L/\tau^2$ one obtains

$$t' < \frac{\pi\tau}{4}\frac{r}{L} \propto \frac{r}{L}\,.$$

Since $r \ll L$ by the problem statement, r/L is small and hence $t' \ll \tau$. Therefore the initial assumption of neglecting t' is justified.

Problem 26 ... falling square — 5 points

A very long, vertically oriented rod of negligible diameter carries a current $I = 1.7\,\mathrm{A}$ *upward. A thin conductive square of side* $a = 15\,\mathrm{cm}$, *linear resistance* $\lambda = 230\,\mathrm{m\Omega\cdot m^{-1}}$, *and mass* $12\,\mathrm{g}$ *is placed symmetrically around the rod and released. Determine the speed of the square after* $7.5\,\mathrm{s}$.

Jarda is trying to stop his things from falling.

The current in the vertical rod generates a magnetic field around it. Its field lines are tangent to concentric circles centered on the rod and lying in planes perpendicular to it. The magnitude of the magnetic induction at a distance r from the rod is

$$B = \frac{\mu_0 I}{2\pi r},$$

where μ_0 is the vacuum permeability and I is the current in the rod.

As the square falls through this field, a voltage is induced along its perimeter according to Faraday's law of electromagnetic induction:

$$U = \oint B_\perp v\,\mathrm{d}l,$$

where $B_\perp$ is the component of the magnetic field perpendicular to each segment of the square of length $\mathrm{d}l$ and v is the velocity of the square relative to the rod. The integral is taken over the entire perimeter.

However, this line integral is zero. On each side of the square, the perpendicular component of the magnetic field points both into and out of the square symmetrically about the center of the side, so all contributions cancel. No net voltage is induced, and the situation is equivalent to the case of no current in the rod.

Therefore, the square falls with constant gravitational acceleration g. Its speed after $t = 7.5$ s is

$$v_t = gt \approx 74 \text{ m/s}.$$

An alternative approach considers the change of magnetic flux through the square. Since the field is always in the plane of the square, the flux through it is zero, and thus its time derivative is also zero. No voltage is induced, and the magnetic field does not affect the motion of the square.

Problem 27 ... ion thrust engine

4 points

There is currently work on the concept of a space drive that accelerates a rocket by ejecting plasma accelerated to relativistic speeds. Consider a rocket with a total mass of 100 t *that we accelerate by ejecting protons. Each proton has a total energy of* 1 TeV *and* 10^{-3} mol *of protons are emitted from the rocket per second. How long will it take to accelerate the rocket to a speed of* 100 km/s *in this way? Assume the rocket's mass remains approximately constant during the acceleration.*

Jirka listened to a very interesting podcast.

Let us denote $\nu = nN_\mathrm{A}$ as the number of protons emitted per second, where $n = 10^{-3}\,\mathrm{mol\cdot s^{-1}}$ and N_A is Avogadro's constant. During a time interval Δt, the accelerated protons leaving the rocket carry a total momentum

$$\Delta p = p_1 \nu \Delta t\,,$$

where p_1 is the momentum of single proton, which we will compute shortly. From the conservation of momentum, the rocket gains the same momentum Δp during time Δt, which corresponds to a force

$$F = \frac{\Delta p}{\Delta t} = \nu p_1\,.$$

The momentum of a single proton p_1 is obtained from the relativistic relation

$$E^2 = E_0^2 + p_1^2 c^2\,,$$

where $E_0 = 938\,\mathrm{MeV}$ is the rest energy of the proton. We see that this is much smaller than the total proton energy 1 TeV, so we may neglect it and express the momentum as

$$p_1 = \frac{E}{c}\,.$$

Let us note that when computing the momentum, we will need to convert the units. This can easily be done using the conversion

$$1eV = 1.602 \cdot 10^{-19}\,\mathrm{J}\,,$$

which follows from the value of the elementary charge e.

The only remaining step is to express the acceleration time. A constant force acts on the rocket, and according to the problem statement, we neglect the change in the rocket's mass (as the resulting time will show, this is a reasonable assumption), so the acceleration remains constant. Thus, the acceleration time is

$$t = \frac{v}{a} = \frac{mv}{F}\,,$$

where $m = 100\,\mathrm{t}$ is the mass of the rocket. Substituting the previous intermediate results, we obtain

$$t = \frac{mvc}{nN_A E} \doteq 3.1 \cdot 10^4\,\mathrm{s} \doteq 8.6\,\mathrm{h}.$$

Problem 28 ... action scene with motorbike and truck

5 points

A block with a mass of $M = 123\,\mathrm{kg}$ moves without friction at a constant velocity of $v = 7.35\,\mathrm{m{\cdot}s^{-1}}$ along a horizontal surface. On top of the block is a cylinder with a mass of $m = 11.0\,\mathrm{kg}$, which rolls with a velocity of $u = 2.23\,\mathrm{m{\cdot}s^{-1}}$ relative to the block in the direction of the block's motion. The cylinder, therefore, eventually rolls off the block and begins to move along the plane as well, assuming that the vertical component of its velocity is absorbed upon impact. How long after hitting the surface will the cylinder collide with the block? Assume that their mutual distance at the moment of impact was negligible. The coefficient of kinetic friction between the cylinder and the plane is $f = 0.278$.

Vašek is already starting to come up with physics problems.

When the cylinder is falling on the plane, the centre-of-mass velocity is $u+v$, whereas the rim points have velocity u, so slipping begins at the contact surface. Friction will indeed increase the cylinder's angular velocity, but the centre-of-mass will decelerate. For an appropriate choice of parameters the cylinder may ultimately move slower than v, allowing the block to catch up.

Because the tangential velocity of rim points relative to the cylinder's centre differs from the centre-of-mass velocity relative to the ground, a frictional force acts between the cylinder and the surface, witch has magnitude of

$$F = mgf\,,$$

where m is the mass of the cylinder, g the gravitational acceleration, and f the coefficient of friction. This force decelerates the cylinder's centre of mass, so that its velocity satisfies

$$v_\mathrm{v} = u + v - at = u + v - gft\,,$$

where $a = F/m$ is the deceleration of the cylinder. The same force produces a torque and thus spins the cylinder up to a larger angular velocity:

$$M = Fr = J\varepsilon\,,$$

where M is the torque, r the cylinder radius (the lever arm), $J = mr^2/2$ the moment of inertia about the rotation axis, and ε is the angular acceleration. The angular velocity evolves as

$$\omega = \omega_0 + \varepsilon t\,.$$

Once the cylinder has been spun up so that the rim points move at the same speed as the cylinder's centre, the frictional force vanishes and the cylinder continues with uniform rectilinear motion. We therefore find the time at which this occurs. At that moment

$$\omega r = \omega_0 r + \varepsilon t r = u + v - gft \quad\Rightarrow\quad \tau = \frac{v}{3gf} \doteq 0.898\,\mathrm{s}\,,$$

where we used $\omega_0 r = u$ and the relations above.

For the block to catch the cylinder the cylinder's final speed must be less than the block's speed, i.e.

$$v > u + v - gf\tau = u + \frac{2v}{3} \quad\Rightarrow\quad v > 3u\,.$$

Substituting the problem values verifies that this condition holds and that a collision will occur.

We now determine the positions of the cylinder and the block relative to the ground as functions of time, taking the point where the cylinder falls off the block as $x = 0$. The block's position is given by

$$x_\mathrm{k} = vt\,.$$

The cylinder's motion is more complex: it first decelerates uniformly and then moves at constant speed. During deceleration it travels a distance of

$$x_\mathrm{v1} = (v+u)\,\tau - \frac{1}{2} g f \tau^2 = \frac{v\,(6u+5v)}{18 g f}$$

and thereafter its position is

$$x_\mathrm{v} = x_\mathrm{v1} + (u + v - g f \tau)\,(t-\tau)\ .$$

Solving for the time t_s at which the two coordinates coincide,

$$\begin{aligned}
x_\mathrm{k} &= x_\mathrm{v}\ ,\\
v t_\mathrm{s} &= x_\mathrm{v1} + (u+v-gf\tau)\,(t_\mathrm{s}-\tau)\ ,\\
t_\mathrm{s} &= \frac{x_\mathrm{v1} - \tau\,(u+v-gf\tau)}{gf\tau - u}\,,\\
t_\mathrm{s} &= \frac{\frac{v(6u+5v)}{18gf} - \frac{v}{3gf}\left(u + \frac{2v}{3}\right)}{\frac{v}{3} - u}\,,\\
t_\mathrm{s} &= \frac{v^2}{18gf}\frac{1}{\frac{v}{3}-u} \doteq 5.00\,\mathrm{s}\,.
\end{aligned}$$

Problem 29 ... water turn

6 points

Consider a boat of mass $m = 900\,\mathrm{kg}$ shaped as a rectangular prism with length $a = 4.0\,\mathrm{m}$, width $b = 1.2\,\mathrm{m}$, and height $c = 80\,\mathrm{cm}$. The boat is propelled by a propeller that delivers an angular momentum of magnitude $L = 400\,\mathrm{J{\cdot}s}$ directed toward the bow. On the water surface, the boat turns with an angular velocity of $10\,^\circ{\cdot}\mathrm{s}^{-1}$ to the right. What is the angle between the water surface and the deck during this maneuver? If the bow is higher than the stern, give a positive value. Assume the center of mass of the boat is at its geometric center.

Jarda would like to ride a flying camel.

The angular momentum of a ship's propeller, due to its rigid attachment to the ship's structure, always points from the stern to the bow in the horizontal plane. When the ship turns, the magnitude of this angular momentum does not necessarily change, but its direction does in the inertial reference frame. To change angular momentum in an inertial frame, one must apply an external torque according to

$$\mathbf{M} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\,.$$

Let us use the analogy with the change of the velocity vector $\mathbf{v}$ during uniform circular motion of a point mass. Here, the acceleration $\mathbf{a}$ satisfies

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{v}\,, \tag{5}$$

where $\boldsymbol{\omega}$ is the angular velocity vector. In the inertial reference frame, the vector $\mathbf{v}$ rotates about the axis of rotation, while in the rotating frame it remains at rest. The centripetal acceleration $\mathbf{a}$ must be provided by some external constraint. Similarly, the angular momentum vector remains unchanged in the rotating reference frame attached to the boat. The torque $\mathbf{M}$ is therefore analogous to the acceleration $\mathbf{a}$ from Eq. (5) , and the angular momentum $\mathbf{L}$ of the propeller is analogous to the velocity $\mathbf{v}$. Introducing the turning angular velocity vector $\boldsymbol{\Omega}$ of the boat, we may write the change of angular momentum as

$$\mathbf{M}_s = \boldsymbol{\Omega} \times \mathbf{L}\,.$$

Because the vector Ω points perpendicular to the water surface and the vector $\mathbf{L}$ points from the stern to the bow, the torque vector $\mathbf{M}_s$ is directed perpendicular to the boat's side, causing it to tilt on the water. Its direction can be determined using Žukovsky's rule:

If a flywheel (in our case, the ship's propeller) is forced to undergo precessional motion, a gyroscopic torque arises that attempts to rotate the axis of the flywheel toward the axis of the imposed motion along the shortest path.[1]

When turning right, the angular velocity vector points downward beneath the water surface. When the angular momentum vector points toward the bow, rotating it downward requires the bow to sink deeper into the water. Thus, during a turn the stern will be higher than the bow, so the sign of the tilt angle will be negative according to the problem statement. We now need to determine its magnitude.

The gyroscopic torque that presses the bow of the boat into the water will be balanced by the reaction torque. We must now find what torque arises when the boat is tilted by an angle α. Assume this angle is small, so we may use the approximations $\alpha \approx \sin\alpha \approx \tan\alpha$. Its magnitude depends on the center of mass of the submerged part and its size. In the untilted state, the draft of the boat h (distance from the bottom to the water surface of density ρ) satisfies

$$V\rho g = mg \quad \Rightarrow \quad h = \frac{V}{ab} = \frac{m}{\rho ab}$$

and the point of application of this force is located in the middle of the submerged section, i.e. at a distance $h/2$ above the bottom. The boat rotates around a point lying on its axis of symmetry and on the water surface. The volume of the submerged part does not change, since the magnitude of the forces remains the same, but the distribution changes, producing a torque. The torque due to gravity is

$$M_g = F_g\left(\frac{c}{2} - \frac{h}{2}\right)\alpha\,,$$

where we expressed the distance of the boat's center of mass from the pivot point.

At the same time, a larger portion of the boat is submerged on one side, forming a right triangle with legs $a/2$ and $(a/2)\tan\alpha$. Since we expect only a small tilt, the horizontal projection of its centroid lies at a distance

[1] Rule taken from `http://fyzikalniolympiada.cz/texty/setrv.pdf`.

$$p_1 = \frac{2}{3}\frac{a}{2}\,,$$

from the axis of rotation, and the corresponding force is

$$F_1 = \frac{1}{2}\frac{a}{2}\frac{a}{2}\alpha b\rho g\,,$$

producing torque

$$M_1 = F_1 p_1 = \alpha b\rho g\frac{a^3}{24}\,.$$

An equal and opposite torque is produced by the emerged triangular section, since the buoyant force there is missing. Thus, the total buoyant torque includes $2M_1$. For a more detailed explanation, we refer the reader to the Physics Olympiad text at `http://fyzikalniolympiada.cz/texty/kapaliny.pdf` .

The total torque is therefore

$$M = 2M_1 - M_g = \alpha b\rho g\frac{a^3}{12} - mg\left(\frac{c}{2} - \frac{h}{2}\right)\alpha = \\ = \frac{g}{2}\left(\frac{1}{6}a^3 b\rho - mc + \frac{m^2}{\rho ab}\right)\alpha\,.$$

We can now compare this torque to the torque needed to change the direction of the angular momentum of the propeller, yielding the desired angle

$$|\mathbf{M}_s| = M \quad \Rightarrow \quad \alpha = \frac{2\Omega L}{g\left(\frac{1}{6}a^3 b\rho - mc + \frac{m^2}{\rho ab}\right)} \doteq 0.067\,^\circ\,.$$

Given the directions of the vectors, the final answer must be negative, so $\alpha = -0.067\,^\circ$.

As an interesting note, the same physical phenomenon was responsible for the famous maneuverability of the Sopwith Camel fighter aircraft in World War I. Its rotary engine carried a large angular momentum. During a right turn, the nose pitched downward, increasing speed and tightening the turn. During a left turn, the nose pitched upward, making left turns sluggish. Experienced pilots used this effect to their advantage in combat.

Problem 30 ... chasing the Sun — 5 points

At 6:10 AM, Danka is sitting in an airplane taking off from Prague Airport, heading directly east. The airplane ascends at an angle of $30\,^\circ$ with a speed of $300\,\mathrm{km{\cdot}h^{-1}}$. On this day, the sunrise in Prague is expected to occur at 6:38 AM, but Danka will witness it earlier from the plane. How many minutes after takeoff will Danka observe the sunrise? For the purposes of this problem, Prague has been moved to the equator, and Danka is flying on the equinox. Also, consider the Earth to be a perfect sphere, with a 24-hour day, and neglect the Earth's motion around the Sun.

Thanks to flying, Danka has experienced an earlier sunrise several times.

We translate the entire problem into the language of analytic geometry. Consider Cartesian axes x and y such that their origin $[0,0]$ is at the center of the Earth. At this moment, the Earth is represented by a circle of radius R described by the equation:

$$x^2 + y^2 = R^2$$

and we rotate the axes so that Prague lies exactly on the y-axis, giving Prague the coordinates $P\,[0,R]$ at time t_0 (6:10 a.m.).

Next, describe the Sun ray representing dawn. This ray is the tangent to the Earth passing through the point where dawn begins on the surface. A tangent to a circle at point $T\,[x_0, y_0]$ with center $V\,[m,n]$ can be written as:

$$(x-m)\,(x_0-m) + (y-n)\,(y_0-n) = R^2\,,$$

which, in our case, simplifies to:

$$xx_0 + yy_0 = R^2\,.$$

Here x_0 and y_0 represent the points on the circle where it is currently dawn. These points can be described by an angle from the y-axis. Knowing that at 6:38 a.m. dawn occurs at $P\,(0,R)$, the current location is at an angle corresponding to 28 minutes of the Sun's movement ($\Delta t = t_P - t_0$, where t_P is the time of dawn in Prague and t_0 is the current time). The angular velocity is:

$$\omega = \frac{2\pi}{24}\ \mathrm{rad\cdot h^{-1}}\,,$$

so the angle between the current dawn location and Prague is:

$$\varphi = \Delta t \cdot \omega = \frac{28}{60}\cdot\frac{2\pi}{24}\,.$$

This point lies on the circle, and its Cartesian coordinates are:

$$\begin{aligned} x_{01} &= R\sin(\varphi)\,, \\ y_{01} &= R\cos(\varphi)\,. \end{aligned}$$

The angle between Prague and the current dawn location decreases over time as the Sun moves. Since the angle decreases at the angular velocity ω, the time-dependent coordinates are:

$$\begin{aligned} x_0 &= R\sin(\varphi - \omega t)\,, \\ y_0 &= R\cos(\varphi - \omega t)\,. \end{aligned}$$

Now, consider the airplane. It departs at t_0 from $P\,(0,R)$ with speed $v = 300\,\mathrm{km\cdot h^{-1}}$ at an angle of 30°. Using right-triangle decomposition, its position at time t is:

$$\begin{aligned} x &= \cos(30^\circ)\,vt\,, \\ y &= R + \sin(30^\circ)\,vt\,. \end{aligned}$$

The airplane will be on the tangent line (the Sun ray) when the points x_0, y_0, x, y satisfy the tangent equation. Substituting the time-dependent positions gives:

$$\begin{aligned} R^2 &= R\sin(\varphi-\omega t)\,(\cos(30^\circ)\,vt) + R\cos(\varphi-\omega t)\,(R+\sin(30^\circ)\,vt) = \\ &= R\sin(\Delta t\cdot\omega-\omega t)\,(\cos(30^\circ)\,vt) + R\cos(\Delta t\cdot\omega-\omega t)\,(R+\sin(30^\circ)\,vt)\,. \end{aligned}$$

This equation is difficult to solve analytically, so we solve it numerically. By subtracting R^2 and plotting the resulting function against t in Geogebra software (Fig. 3), we find the intersection with the x-axis. The only physically meaningful solution (non-negative and less than 28/60 hours) is $t \approx 0.1376$ h ≈ 8.3 min. Thus, Danka will see the sunrise approximately 8 minutes after takeoff, i.e., at 6:18 a.m.

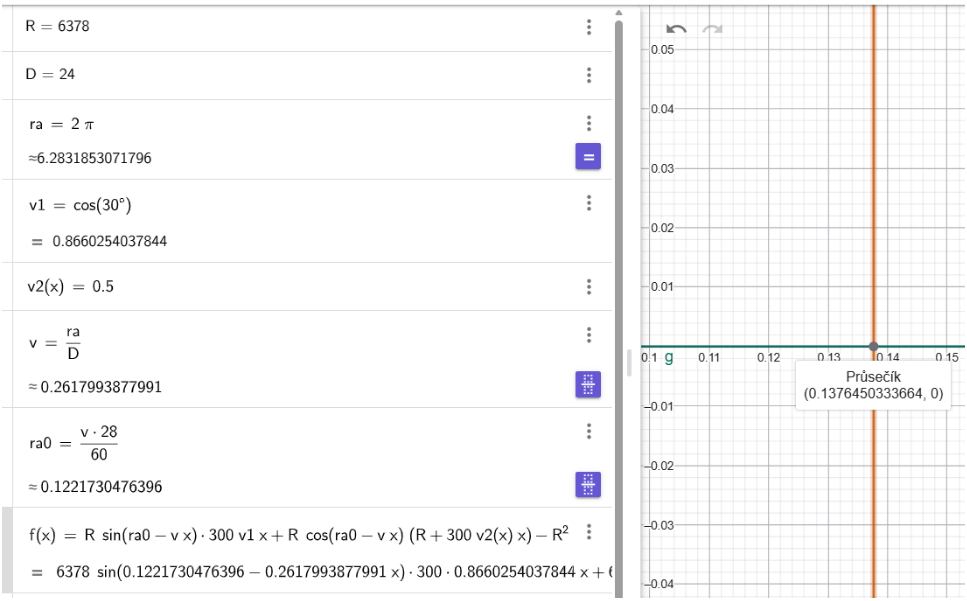
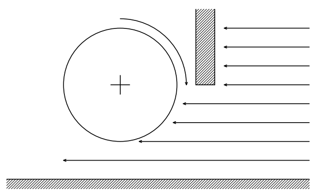


Figure 3: Numerical solution of the equation from Geogebra

Problem 31 ... gas flow 5 points



Consider a cylinder of radius $r = 3.00\,\mathrm{mm}$ *and height* $h = 5.00\,\mathrm{mm}$, *which can rotate about its axis of symmetry. Gas is directed at the cylinder so that particles strike only one half of the cylinder. The other half is covered (the boundary between the halves is the plane that contains the cylinder's symmetry axis, see the figure). The cylinder spins up to a frequency* $f = 65.0\,\mathrm{Hz}$. *Assume that nothing strikes the cylinder from the other side. A resistive torque* $M = 2.00 \cdot 10^{-9}\,\mathrm{N{\cdot}m}$ *acts on its rotation such that upon impact, a gas molecule changes its velocity to the velocity of the cylinder surface at that point. The gas flows uniformly across the cross-section that hits the cylinder. What amount of substance of gas (in mol per second) strikes the cylinder? The gas is argon with a volumetric concentration* $c = 2.00 \cdot 10^{-4}\,\mathrm{mol \cdot m^{-3}}$.

Jarda likes to direct gas streams at objects during his free time.

Let the particle velocity before impact be denoted by v. When a particle hits the cylinder at a perpendicular distance x from the axis, the angular momentum it transfers to the cylinder is

$$\Delta L = mvx - 2\pi fmr^2 \,,$$

where m is the mass of the molecules. The first term is, in absolute value, the angular momentum the molecule had with respect to the cylinder axis before the collision; the second term is the angular momentum after the collision, so their difference gives the angular momentum transferred to the cylinder. Note that for some molecules $\Delta L < 0$ (for example, molecules that hit the front of the cylinder, for which $x = 0$ clearly gives $\Delta L < 0$), and in that case the molecules simply slow the cylinder down (which makes sense).

If the molecular flux is homogeneous across the cross-section, then all $d \in (0, r)$ are equally probable, i.e., they have probability density $1/r$. Therefore, the average molecule transfers angular momentum

$$\Delta \bar{L} = \frac{1}{r}\int_0^r \left(mvx - 2\pi fmr^2\right) \mathrm{d}x = \frac{1}{2}mvr - 2\pi fmr^2 \,.$$

(This really didn't need to be integrated — one only needs to note that the second term is constant and the first term has its "center of mass" at $r/2$.)

The problem asks how many moles of molecules per second strike the cylinder. Let this unknown be $\dot{n}$. Then the torque exerted by the molecules on the cylinder is equal to the angular momentum transferred per unit time:

$$M = \Delta \bar{L} N_\mathrm{A} \dot{n} = (\frac{1}{2}vr - 2\pi fr^2) M_\mathrm{Ar} \dot{n} \,,$$

where M_Ar is the molar mass of argon. At the same time, we can note that the cylinder's rotation frequency settles at the value for which this torque equals the resistive torque given in the problem, so we already have an equation for $\dot{n}$.

The only remaining unknown in that equation is the molecular velocity, which we do not yet know. Here we must reason by analogy with mass and volumetric flow rates and realize that

$$\dot{n} = cvS \,,$$

where c is the volumetric concentration of molecules (given) and S is the cross-section of the gas stream, which in this case is $S = rh$. Thus $v = \dot{n}/(crh)$; substituting this into the equation and solving for $\dot{n}$ gives $\dot{n}$

$$0 = \dot{n}^2 \frac{1}{2ch} - \dot{n} 2\pi fr^2 - \frac{M}{M_\mathrm{Ar}}$$

$$\dot{n} = ch\left(2\pi fr^2 \pm \sqrt{(2\pi fr^2)^2 + \frac{2M}{chM_\mathrm{Ar}}}\right) \,,$$

where the minus root would be negative and therefore physically meaningless (the molecules would have to bounce off the cylinder with velocity v directly into the nozzle). The answer is thus

$$\dot{n} = ch\left(2\pi fr^2 + \sqrt{(2\pi fr^2)^2 + \frac{2M}{chM_\mathrm{Ar}}}\right) \doteq 3.20 \cdot 10^{-7}\,\mathrm{mol{\cdot}s^{-1}} \,,$$

Problem 32 ... warming up by work 5 points

We have a piston filled with air (i.e., the Poisson's constant is equal to $\kappa = 1,4$) under standard conditions. The surroundings are also at standard conditions, and the piston walls are thermally conductive. We want to bring the air to a state where it has twice the temperature (in the thermodynamic sense) and the same volume as it has now. We have a heater that can supply the necessary heat; however, we want to conserve the heater and its energy for tough times. Therefore, we aim to partially supply the energy to the air through mechanical work. Specifically, we can expand the piston volume to twice its current volume (and then compress it back), and we can use the heater at any point during this process. What is the minimum amount of energy we must supply via the heater? Express your answer as a percentage of the amount we would need to provide if we did not perform any mechanical work and only used a heater. *Lego remembered inexact differentails used in thermodynamics.*

First, we pull the piston out to a volume of $2V$. The amount of work we perform in doing so, and how much heat we obtain from the surroundings, depends on the speed of this expansion. However, that does not matter in this problem, because we are concerned only with the heat from the heater, and in this step, it is zero in any case. We then wait until the temperature of the air in the piston equalizes with the ambient temperature, i.e., becomes T; again, we have not used any heat from the heater.

Now the speed begins to matter: if we compressed the air isothermally, we would return to the original state, but that would not be beneficial. So we compress it infinitely fast (adiabatically), so that it does not lose any energy through heat exchange. For an adiabatic process, among other things, $TV^{\kappa-1} = \mathrm{const}$ holds, so after this compression the temperature will be

$$T_1 V^{\kappa-1} = T(2V)^{\kappa-1} \quad \Rightarrow \quad T_1 = T2^{\kappa-1} \,.$$

As a result, we must use the heater to supply the gas with energy equal to the difference between the internal energy of the gas in this state and at the temperature $2T$, i.e.

$$Q_1 = c_V N(2T) - c_V NT2^{\kappa-1} = (2 - 2^{\kappa-1})c_V NT \,,$$

where c_V is the heat capacity of air at constant volume (we could express it using the Poisson constant κ), and N is the number of molecules in the piston (which we can obtain from p_A, V, T). However, the same factors appear in the case where we perform no work, so they cancel out in the end.

What if it were more advantageous to heat the air while it is still expanded, and only then compress it? Let T_2 be the temperature to which we must heat it so that after compressing back to V, it has the temperature $2T$. Then this temperature is

$$2TV^{\kappa-1} = T_2(2V)^{\kappa-1} \quad \Rightarrow \quad T_2 = T2/2^{\kappa-1} = T2^{2-\kappa} \,.$$

This means that the heater had to heat the air with the energy

$$Q_2 = c_V NT2^{2-\kappa} - c_V NT = (2^{2-\kappa} - 1)c_V NT \,,$$

where c_V and N (and T) are the same as for Q_1, so we only need to compare the parentheses. We can simply substitute the κ from the problem, but it is highly recommended to plot the brackets as a function of κ, which reveals that for $\kappa \in (1, 2)$ we have $Q_2 < Q_1$, and in reality $\kappa \in (1, 5/3) \subset (1, 2)$. Thus, it follows that it is more advantageous to heat the gas first and only then compress it back.

Someone might object that there are infinitely many other possibilities, in which we distribute the heating into several moments arbitrarily placed during the compression. However, it should be intuitive that the two options we examined are extreme cases from an infinite set of possibilities, and a local extremum should lie in one of these extremes of this infinite-dimensional interval. The physical reason why heating the gas fully at the beginning is optimal can be nicely seen, for example, in the fact that doing so increases the pressure in the piston, and therefore, more work is done during the subsequent compression. The total energy we supply through work during compression is the energy we save for the heater. Any "delay in heating" only reduces the work done during compression, thereby increasing the energy the heater must supply.

The minimal energy that must be supplied by the heater is therefore Q_2. The answer is to be given as a percentage of the energy that would be needed without performing any work. In that case, the heater would have to supply

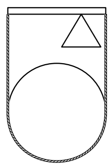
$$Q_0 = c_V N(2T) - c_V NT = c_V NT\,,$$

so our work reduces the required heat to

$$\frac{Q_2}{Q_0} = \frac{(2^{2-\kappa} - 1)c_V NT}{c_V NT} = (2^{2-\kappa} - 1) \doteq 52\,\%$$

of the baseline value.

Problem 33 ... lever–pulleys 5 points



We have a pulley and a lever connected by a massless rope as shown in the figure. Treat the lever as a rod of length $2r = 20\,\mathrm{cm}$ with mass $m = 300\,\mathrm{g}$. The lever is supported so that the lengths of its arms are in the ratio $3:1$. The pulley has the shape of a disk of radius $r = 10\,\mathrm{cm}$ and mass $M = 600\,\mathrm{g}$. Both the lever and the pulley are homogeneous and the rope does not slip on the pulley. What is the acceleration of the center of mass of the pulley at the moment of release from the state shown in the figure?

Lego has never tried to combine these simple machines together.

Let us denote the tension in the rope on the left side as T_1 and on the right side as T_2. With respect to the support point, the lever is subjected to a torque $M_\mathrm{p} = 3rT_1/2 - rT_2/2 + rmg/2$. The moment of inertia of the lever is obtained from Steiner's theorem

$$I_\mathrm{p} = I_\mathrm{p0} + I_\mathrm{pS} = \frac{1}{12}m(2r)^2 + \left(m\frac{r}{2}\right)^2 = \frac{7}{12}mr^2\,.$$

The angular acceleration of the lever is then

$$\varepsilon_p = \frac{M_\mathrm{p}}{I_\mathrm{p}} = \frac{\frac{3}{2}rT_1 - \frac{1}{2}rT_2 + \frac{1}{2}rmg}{\frac{7}{12}mr^2} = \frac{6}{7}\frac{3T_1 - T_2 + mg}{mr}\,,$$

where we define the positive direction as that in which the left side of the lever moves downward (the intuitively expected direction of motion).

The torque acting on the pulley with respect to its center is $M_\mathrm{k} = r(T_2 - T_1)$ (the positive direction again corresponds to the left side of the lever moving downward). The moment of inertia of the disk is $I_\mathrm{k} = Mr^2/2$, then its angular acceleration is

$$\varepsilon_\mathrm{k} = \frac{M_\mathrm{k}}{I_\mathrm{k}} = \frac{r(T_2 - T_1)}{\frac{1}{2}Mr^2} = 2\frac{T_2 - T_1}{Mr}\,.$$

The net force acting on the pulley is $F_\mathrm{k} = Mg - T_1 - T_2$ (positive direction downward), so the acceleration of its center of mass is

$$a_\mathrm{k} = \frac{F_\mathrm{k}}{M} = g - \frac{T_1 + T_2}{M}\,.$$

We have obtained all equations of motion. Now let us move to the geometry of the problem. Imagine the lever rotates by an angle $\mathrm{d}\varphi_\mathrm{p}$. Then on the left side the rope moves downward by $3/2r\,\mathrm{d}\varphi_\mathrm{p}$ and on the right side upward by $1/2r\,\mathrm{d}\varphi_\mathrm{p}$. If we draw a horizontal reference line sufficiently below the pulley, then after this rotation there will be a total of $r\,\mathrm{d}\varphi_\mathrm{p}$ more rope below it. We divide this between the two sides of the pulley, and therefore the center of mass of the pulley descends by $\mathrm{d}x_\mathrm{k} = 1/2r\,\mathrm{d}\varphi_\mathrm{p}$.

At the same time, on the right side the lever rose by $1/2r\,\mathrm{d}\varphi_\mathrm{p}$ and the pulley descended by $1/2r\,\mathrm{d}\varphi_\mathrm{p}$, so between them there is now $r\,\mathrm{d}\varphi_\mathrm{p}$ more rope than before. Since the rope and pulley do not slip, the pulley must have rolled along the rope by exactly this length. That means it must have rotated by an angle

$$\mathrm{d}\varphi_\mathrm{k} = \frac{r\,\mathrm{d}\varphi_\mathrm{p}}{r} = \mathrm{d}\varphi_\mathrm{p}\,.$$

These two relationships were derived for infinitesimal displacements, but differentiating twice with respect to time gives equivalent relationships for accelerations:

$$\begin{aligned} a_k &= 1/2r\varepsilon_\mathrm{p} \\ \varepsilon_\mathrm{k} &= \varepsilon_\mathrm{p}\,. \end{aligned}$$

We now have 5 equations (3 equations of motion + 2 geometric relationships) and 5 unknowns $(\varepsilon_p, \varepsilon_\mathrm{k}, a_\mathrm{k}, T_1, T_2)$, which is solvable. We begin with

$$\begin{aligned} \varepsilon_\mathrm{k} &= \varepsilon_\mathrm{p} \\ 2\frac{T_2 - T_1}{Mr} &= \frac{6}{7}\frac{3T_1 - T_2 + mg}{mr} \\ 7m(T_2 - T_1) &= 3M(3T_1 - T_2 + mg) \\ \frac{(7m + 3M)T_2 - 3Mmg}{9M + 7m} &= T_1\,. \end{aligned}$$

Next we use the relationship between a_k and ε_p, but since the expression would be long, we directly use the fact that the angular accelerations are equal and write

$$\begin{aligned} a_\mathrm{k} &= \frac{1}{2} r \varepsilon_\mathrm{k} \\ g - \frac{T_1 + T_2}{M} &= \frac{1}{2} r 2 \frac{T_2 - T_1}{Mr} \\ Mg - T_1 - T_2 &= T_2 - T_1 \\ \frac{Mg}{2} &= T_2 \,. \end{aligned}$$

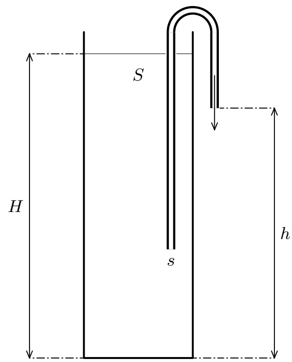
We substitute the previous results into the equation of motion for the acceleration:

$$a_\mathrm{k} = g - \frac{T_1 + T_2}{M} = g - \frac{(7m + 3M)\frac{g}{2} - 3mg}{9M + 7m} - \frac{g}{2} = g \frac{9M + 7m - (7m + 3M) + 6m}{2(9M + 7m)}$$

$$a_\mathrm{k} = g \frac{6M + 6m}{2(9M + 7m)} = 3 \frac{M + m}{9M + 7m} g = 3.5\,\mathrm{m \cdot s^{-2}} \,.$$

Problem 34 ... a straw

6 points



Jarda was playing with his cocktail. He took a straw and fully submerged it in the cocktail. Then he pulled out one end, and to keep the liquid inside, he held the pulled-out end with his finger so that the cocktail would not spill out. Then he released it, and the cocktail began to spill onto the table, much to the displeasure of everyone sitting around.

Calculate how long this calamity will last if Jarda does not intervene. We know that the horizontal cross-section of the glass is $S = 16\,\mathrm{cm^2}$ and the straw has a constant cross-section $s = 0.40\,\mathrm{cm^2}$ along its entire length. Initially, the liquid level in the glass is at a height $H = 8.0\,\mathrm{cm}$ from the bottom, and the straw's outlet is outside the glass at a height $h = 5.0\,\mathrm{cm}$ above the bottom of the glass.

Assume that the cocktail is an ideal incompressible fluid. Neglect any undesirable effects, such as bubble intrusion or surface tension. The straw does not move while the liquid flows out.

Jarda has a remarkable ability of spilling anything he tries to drink.

Before we proceed to the solution, let us make the following consideration. The cocktail will flow out of the straw only when its surface lies above the straw's outlet. If this were not the case—i.e., if the cocktail flowed even when it were below the outlet—this would be equivalent to the liquid flowing uphill. But we do not really observe such behavior in the real world. Thus, this problem is equivalent to the one where the cocktail flows from an opening in the glass located at height h above the bottom.

We solve such a problem using two laws: the continuity equation and Bernoulli's equation. Once the cocktail is flowing out, the liquid level will drop. We describe the height of the level by a coordinate $y > 0$. Let us denote the magnitude of the rate of decrease of the level by V, and note that it must be a function of the level height, i.e., $V = V(y)$. We denote the outflow velocity of the cocktail at the straw's orifice by v, and again note that $v = v(y)$. Thus we may write the continuity equation

$$sv = SV \quad \Rightarrow \quad v = \frac{S}{s}V\,. \tag{6}$$

Now let us write Bernoulli's equation

$$\frac{1}{2}\rho v^2 + \rho hg + p = \frac{1}{2}\rho V^2 + \rho yg + p\,. \tag{7}$$

We substitute (6) into (7) and cancel p. We solve for the velocity V. After rearranging, we obtain

$$V(y) = \sqrt{\frac{2g(y-h)}{\left[\left(\frac{S}{s}\right)^2 - 1\right]}}\,.$$

Now, recall that the velocity $V(y)$ describes the temporal change of the liquid level. Observe, however, what the velocity vector V looks like. In arbitrary Cartesian coordinates we may express it as $\vec{V} = (0, -V(y), 0)$, where the negative sign appears because the velocity expresses a decrease. (Since we considered $y > 0$ upward from the level and the velocity vector points toward the level, the sign must be opposite.) Thus we have $V(y) = -\dot{y}$, which yields the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\sqrt{\frac{2g(y-h)}{\left[\left(\frac{S}{s}\right)^2 - 1\right]}}\,.$$

With the keen eye of an experienced mathematician, we see that dividing by $\sqrt{y-h}$ gives us a differential equation with separated variables. Integrating, we obtain

$$\int_H^h \frac{1}{\sqrt{y-h}}\,\mathrm{d}y = -\sqrt{\frac{2g}{\left[\left(\frac{S}{s}\right)^2 - 1\right]}}\int_0^{t_{\max}}\mathrm{d}t\,.$$

The integral on the right-hand side is trivial. We compute the integral on the left-hand side as

$$\int_H^h \frac{1}{\sqrt{y-h}}\,\mathrm{d}y = \left|\begin{matrix} u = y-h & \mathrm{d}u = \mathrm{d}y \\ u(h) = 0 & u(H) = H-h \end{matrix}\right| = \int_{u(H)}^{u(h)} u^{-\frac{1}{2}}\,\mathrm{d}u =$$
$$= 2\left[\sqrt{u}\right]_{H-h}^{0} = 2(0 - \sqrt{H-h}) = -2\sqrt{H-h}\,,$$

which yields

$$-2\sqrt{H-h} = -\sqrt{\frac{2g}{\left[\left(\frac{S}{s}\right)^2 - 1\right]}}\,t_{\max}\,,$$

from which we obtain the final result in the form

$$t_{\max} = 2\sqrt{\frac{\left[\left(\frac{S}{s}\right)^2 - 1\right](H-h)}{2g}} \doteq 3.1\,\mathrm{s}.$$

Problem 35 ... waging a war with windmills 6 points

Don Quixote calmly rides on his faithful Rocinante through Mancha, when suddenly he spots a four-armed giant on the horizon, waving threateningly at him with all four limbs. However, we, who have our wits about us, clearly see that it is not a giant, but a windmill. It has four blades with a rectangular cross-sectional area $S = 300\,\mathrm{cm}^2$ and surface area $P = 24\,\mathrm{m}^2$, which are placed on rigid rods at a distance $d = 1.00\,\mathrm{m}$ from the axis of rotation of the mill and have length $L = 6.00\,\mathrm{m}$. They are made of homogeneous material with Young's modulus $E = 1.00\,\mathrm{MPa}$ and each has mass $M = 170\,\mathrm{kg}$. The "giant's threatening" waving was caused by a gust of wind, which spun the blades to an angular velocity $\omega = 1.60\,\mathrm{s}^{-1}$. To what length L' do the blades elongate? Assume that these are small deformations and they occur only in the radial direction; also assume that the width and thickness of the blades are small compared to the distance from the axis and thus that the magnitude of the centrifugal force depends only on the perpendicular distance from the end of the blade. Do not consider the influence of gravitational force on the deformation of the blades.

Petr was reminiscing about the Mechanics course.

Consider the centrifugal force f acting on a "slice" of a blade of thickness $\mathrm{d}x'$. The mass of this slice is

$$\mathrm{d}m = \rho S\,\mathrm{d}x',$$

where the density can be expressed, due to homogeneity, as

$$\rho = \frac{M}{LS}.$$

For this element, the centrifugal acceleration is $a = \omega^2 r$, where r is the distance from the axis, giving

$$f = \frac{M\omega^2}{L}(d + x')\,\mathrm{d}x',$$

with the coordinate x' measured from the blade's attachment point.

Since no part of the blade detaches in our scenario, the centrifugal force is balanced by an equal centripetal force. By Newton's third law, the element $\mathrm{d}x'$ experiences an equal and opposite force; iterating this argument shows that any element closer to the axis feels a force contribution $-f$ from all slices further out. The total "external" force on an element at x is thus the integral over all elements farther from the axis (i.e., in $(x, L]$):[2]

$$F = \int_x^L \frac{M\omega^2}{L}(d + x')\,\mathrm{d}x' = \frac{M\omega^2}{L}(L - x)\left(d + \frac{L + x}{2}\right).$$

Applying Hooke's law, the strain is

$$\varepsilon = \frac{\sigma}{E} = \frac{M\omega^2}{LES}(L - x)\left(d + \frac{L + x}{2}\right).$$

The total elongation ΔL is the integral of ε over the blade's length:

[2] For further purposes, the positive force direction is taken away from the axis.

$$L' = L + \Delta L = L + \int_0^L \frac{M\omega^2}{LES}\left(Ld + \frac{L^2}{2} - xd - \frac{x^2}{2}\right)\mathrm{d}x = L\left[1 + \frac{M\omega^2}{ES}\left(\frac{d}{2} + \frac{L}{3}\right)\right].$$

Substituting the given values gives

$$L' \doteq 6.22\ \mathrm{m}.$$

Problem 36 ... conducting corner 7 points

The boundary of the first octant of the Cartesian coordinate system forms a conducting corner (i.e., it consists of three quarter-planes, each oriented along one of the x, y, and z directions). A charge q moves along the half-line $x = y = z > 0$. Determine the ratio of the charge Q placed fixed at the origin to the charge q such that the conducting corner can be replaced by Q while the force acting on q remains unchanged. You may neglect the induced magnetic field. *Kuba was sent to the corner as a punishment.*

Let the charge be at the point $\mathbf{r} = a(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z) \equiv a\mathbf{e}$, with $a > 0$. We seek a distribution of charges in space that produces zero potential on the conducting corner.

Using the fact that the plane of symmetry between two opposite charges has constant potential, we construct a configuration invariant under reflection (with sign change) across the planes xy, yz, and xz. All charges are placed along the axes of the individual octants at equal distances from the origin. It can be seen that in octants II, IV, and V we place $-q$, in octants III, VI, and VIII we place $+q$, and in octant VII we place $-q$. Including the original charge in octant I, this yields four positive and four negative charges at the vertices of an imaginary cube of edge $2a$.

Such a configuration indeed produces a constant potential on the conducting corner. In the first octant, only the original charge remains, satisfying all requirements. By the uniqueness of the solution for the given boundary conditions, this is the only correct arrangement.

Each charge generates a Coulomb electric field. The field at the original charge's position due to the seven other charges is

$$\begin{aligned}\mathbf{E}_{\mathrm{im}}(\mathbf{r}) &= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{4a^2}\mathbf{e} + \frac{1}{8a^2}\frac{\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}} + \frac{1}{8a^2}\frac{\mathbf{e}_y + \mathbf{e}_z}{\sqrt{2}} + \frac{1}{8a^2}\frac{\mathbf{e}_z + \mathbf{e}_x}{\sqrt{2}} - \frac{1}{12a^2}\frac{\mathbf{e}}{\sqrt{3}}\right] \\ &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{4} + \frac{1}{4\sqrt{2}} - \frac{1}{12\sqrt{3}}\right)\frac{\mathbf{e}}{a^2} = -\frac{1}{4\pi\varepsilon_0}\frac{qk}{a^2}\mathbf{e},\end{aligned}$$

where

$$k = \frac{1}{4} - \frac{1}{4\sqrt{2}} + \frac{1}{12\sqrt{3}}.$$

The field produced by these fictitious charges is purely Coulombic. By symmetry, any magnetic field generated by the time-varying electric field must also point along $\mathbf{e}$, so it does not affect the motion of the charge, which moves along $\mathbf{e}$. Therefore, the total force can be simulated by placing a single charge Q at the origin. Setting

$$\frac{1}{4\pi\varepsilon_0}\frac{Q}{3a^2} = -\frac{1}{4\pi\varepsilon_0}\frac{qk}{a^2}\|\mathbf{e}\| \quad \Rightarrow \quad Q = 3\sqrt{3}qk \approx -0.630q\,.$$

Problem 37 ... FTL

6 points

Consider an empty container formed by rotating the function $y = x^{-3}$ around the axis $x = 0$ for positive values of y, with all dimensions in meters. The container is closed at the bottom by the rotated x-axis. Water flows into the container at a rate of $Q = 200\,\mathrm{l{\cdot}s^{-1}}$. Assuming the water instantly forms a flat surface, after how long would the water level begin to rise at a superluminal speed?

Marek wanted to be a buoy.

Let us compute the volume of water V in the container when it is filled to a height h. Because the problem is rotationally symmetric, the easiest way to obtain the volume is to integrate in cylindrical coordinates. If the water level is at height $y = h$, then according to the problem statement, the radius of the container at that height must be $r = h^{-1/3}$. Thus, in coordinates (r, φ, z),

$$V = \int_0^{2\pi}\int_0^h\int_0^{z^{-1/3}} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\varphi = 2\pi\int_0^h\int_0^{z^{-1/3}} r\,\mathrm{d}r\,\mathrm{d}z = 2\pi\int_0^h \frac{z^{-2/3}}{2}\,\mathrm{d}z = 3\pi h^{1/3}\,.$$

We can rewrite this as

$$h = \left(\frac{V}{3\pi}\right)^3\,.$$

Now differentiate both sides of the equation with respect to time, obtaining

$$\frac{3V^2}{(3\pi)^3}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}t}\,.$$

We know how the volume of water changes in time – it is the inflow rate Q. At the moment of interest, we also know the rate of change of the water height; it is the speed of light c. Rearranging yields

$$V^2 = \frac{9\pi^3 c}{Q}\,,$$

$$V = 3\sqrt{\frac{\pi^3 c}{Q}}\,.$$

This is the volume of water at the time when the water level rises at the speed of light. How long does it take for this to happen? The inflow rate is constant, so the desired time t is

$$t = \frac{V}{Q} = 3\sqrt{\frac{\pi^3 c}{Q^3}} \approx 3.23\cdot 10^6\,\mathrm{s}\,.$$

Problem 38 ... speed of a shadow

6 points

Jarda played tennis in the sun, which was 32° above the horizon. He served the ball to his opponent by hitting it from a height of 70 cm at an angle of 38° to the ground. The ball landed on the other side of the court at a distance of 25 m from Jarda. Determine the maximum speed of the shadow that the ball casts on the ground. The plane of the ball's trajectory is tilted at an angle of 23° with respect to the vertical plane in which the sun's rays strike the ball.

When Jarda's forehand stopped working, he unleashed physics on his opponent.

First, from the given values, we compute how the ball's x and y coordinates depended on time and its initial velocity. From the equations for projectile motion we obtain

$$x = v_0 \cos \alpha t\,,$$
$$y = h + v_0 \sin \alpha t - \frac{1}{2} g t^2\,,$$

where v_0 is the initial velocity of the ball, α is the launch angle relative to the ground, and h is the initial height of the ball. From the first equation we express t, substitute it into the second, and apply the condition that for $y = 0\,\mathrm{m}$ the horizontal distance of the ball from Jarda must have been $x_{\max} = 25\,\mathrm{m}$. This yields for v_0

$$0 = h + x_{\max} \tan \alpha - \frac{1}{2} g \frac{x_{\max}^2}{v_0^2 \cos^2 \alpha}\,,$$
$$v_0^2 = \frac{g x_{\max}^2}{2\left(h + x_{\max} \tan \alpha\right) \cos^2 \alpha}\,.$$

Next, we consider how the shadow of the ball moves along the ground during this motion. At every moment, rays strike the ball at an angle $\beta = 32$ř, so the shadow will always make an angle β with the ground. However, relative to the direction of the ball's motion, this shadow is deflected by an angle $\gamma = 23$ř. From these data we can compute that the distance of the ball's shadow from the point located directly below it is

$$d = \frac{y}{\tan \beta}\,.$$

The position of the shadow is described by two coordinates. We orient the x-axis along the direction of the ball's flight; the distance of the shadow along this axis is therefore

$$x_s = x - d \cos \gamma\,.$$

We orient the z-axis perpendicular to the x and y axes (we choose the positive direction so that the z-component of the position is positive), and the z-component of the shadow's coordinates is

$$z_s = d \sin \gamma\,.$$

From this we obtain the time dependence of the shadow's coordinate components

$$x_s = v_0 \cos \alpha t - \frac{h + v_0 \sin \alpha t - \frac{1}{2} g t^2}{\tan \beta} \cos \gamma\,,$$
$$z_s = \frac{h + v_0 \sin \alpha t - \frac{1}{2} g t^2}{\tan \beta} \sin \gamma\,.$$

We then differentiate both equations with respect to time to determine the components of the shadow's velocity

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = v_0\cos\alpha - \frac{\cos\gamma}{\tan\beta}\left(v_0\sin\alpha - gt\right)\ ,$$
$$\frac{\mathrm{d}z_s}{\mathrm{d}t} = \frac{\sin\gamma}{\tan\beta}\left(v_0\sin\alpha - gt\right)\ .$$

To determine the total velocity of the shadow, we square both equations and add them. In this way we obtain the square of the shadow's velocity, which we will maximize

$$v^2 = \left(\frac{\sin\gamma}{\tan\beta}\left(v_0\sin\alpha - gt\right)\right)^2 + \left(v_0\cos\alpha - \frac{\cos\gamma}{\tan\beta}\left(v_0\sin\alpha - gt\right)\right)^2 .$$

We differentiate the squared velocity with respect to time and set the derivative equal to zero in order to find the maximum

$$0 = 2\left(\frac{\sin\gamma}{\tan\beta}\left(v_0\sin\alpha - gt\right)\right)\left(-\frac{\sin\gamma}{\tan\beta}g\right) + 2\left(v_0\cos\alpha - \frac{\cos\gamma}{\tan\beta}\left(v_0\sin\alpha - gt\right)\right)\left(\frac{\cos\gamma}{\tan\beta}g\right)\ ,$$

from which we obtain

$$t_{\max} = \frac{v_0\left(\sin\alpha - \tan\beta\cos\alpha\cos\gamma\right)}{g}\ .$$

To verify that we indeed found the time at which the shadow's velocity is maximal, we compute the second derivative of the previous expression with respect to time. We obtain

$$\frac{\mathrm{d}^2v^2}{\mathrm{d}t^2} = 2\frac{g^2}{\tan^2\beta}\left(\sin^2\gamma + \cos^2\gamma\right) = 2\frac{g^2}{\tan^2\beta}\ ,$$

which is certainly greater than zero due to the square term. Thus, we did not find a maximum but a minimum of the velocity. It follows that the shadow attains its maximum velocity either immediately before impact or immediately after launch.

From the equation for the ball's height as a function of time, after substituting $y = 0\,\mathrm{m}$ (its value at impact), we solve for the impact time t_d as

$$t_d = \frac{v_0\sin\alpha + \sqrt{v_0^2\sin^2\alpha + 2gh}}{g}\ .$$

Substituting the given values into this equation and into the equation for the initial velocity v_0 gives $v_0 = 15.62\,\mathrm{m\cdot s^{-1}}$, $t_d = 2.03\,\mathrm{s}$. We can now evaluate the velocity of the shadow by inserting $t = 0\,\mathrm{s}$ for the launch velocity v_1 into

$$v_1 = v_0\sqrt{\cos^2\alpha + \frac{\sin^2\alpha}{\tan^2\beta} - 2\frac{\cos\alpha\cos\gamma\sin\alpha}{\tan\beta}} = 6.3\,\mathrm{m\cdot s^{-1}}\ ,$$

and substituting for $t = t_d$ for the shadow's velocity at impact v_2

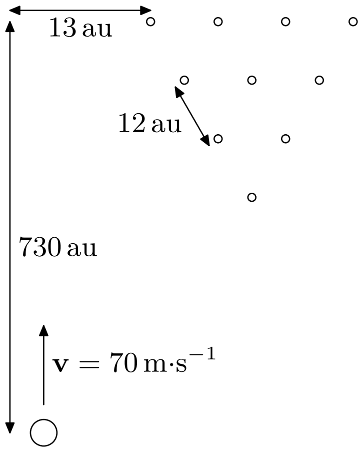
$$v_2 = \sqrt{\left(\frac{\sin\gamma}{\tan\beta}\left(v_0\sin\alpha - gt_d\right)\right)^2 + \left(v_0\cos\alpha - \frac{\cos\gamma}{\tan\beta}\left(v_0\sin\alpha - gt_d\right)\right)^2} = 28\,\mathrm{m{\cdot}s^{-1}}\ .$$

We see that $v_2 > v_1$, and therefore the maximum velocity of the shadow is $28\,\mathrm{m{\cdot}s^{-1}}$.

Problem 39 ... heavenly bowling

7 points

It has been a while since God created the world, and because sometimes He has nothing to do, he likes to play bowling with planets. On the Milky Way, he places ten small planets in a classical equilateral triangle, so that the relative distance between two planets on the edge is 12 au*. For the ball, God choses a star the size of our sun, which he throws in the plane of the "pins" from a distance of* 730 au *from the back row of planets toward them. Although he gave his sphere the necessary rotation, he forgot that there is no friction in space. The star thus did not turn toward the planets, but instead flew parallel to the axis of the original triangle at a distance of* 13 au *from one of its corners at a speed of* $70\,\mathrm{m{\cdot}s^{-1}}$*. Nevertheless, the star did strike some of the planets. How many planets did God "knock off" in this way? The planets are so small that their gravitational interaction is negligible. Also, do not consider the effect they have on the star's orbit.*



Jarda would like to throw like a god.

When the star passes near the location where the planets are positioned, due to mutual gravitational interaction, the planets begin to move towards the star, and it is therefore possible that they collide with it. This is then counted as a contact between a planet and the star. Because the planets are small, we neglect their mutual gravitational interactions, as well as any influence on the star's trajectory caused by their presence, as the problem statement leads us to do.

We will determine the condition under which a planet, initially at rest, collides with a star which is flying by. For this, let us switch to an inertial reference frame in which the star is at rest, and the planet has an initial velocity of $-v$, with an impact parameter b determined by its position in the triangle. In this system, the conservation of energy and the conservation of angular momentum still hold.

The energy of the planet per unit mass is

$$E = \frac{1}{2}v^2 - \frac{M_\odot G}{\sqrt{L^2 + b^2}}\,,$$

where $M_\odot$ is the mass of the star and L is the initial distance between the planet and the line of approach.

The angular momentum of the planet per unit mass is

$$l = bv\,.$$

Let r denote the closest distance of the planet to the star, and u its velocity in this point. From the conservation laws, we have

$$\frac{1}{2}v^2 - \frac{M_\odot G}{\sqrt{L^2 + b^2}} = \frac{1}{2}u^2 - \frac{M_\odot G}{r}$$

and

$$bv = ru\,,$$

since the velocity and radius vectors are perpendicular at perihelion. From these two equations, we can express r as

$$r = \frac{-M_\odot G \pm \sqrt{M_\odot^2 G^2 + b^2 v^2 \left(v^2 - \frac{2M_\odot G}{\sqrt{L^2+b^2}}\right)}}{v^2 - \frac{2M_\odot G}{\sqrt{L^2+b^2}}} \,.$$

If this distance is smaller than R, a collision between the planet and the star occurs. Now we substitute for the distances and determine how many planets touched the star. Since

$$v^2 < \frac{2M_\odot G}{\sqrt{L^2+b^2}} \,,$$

the result is always positive (independent of the sign in the equation). However, from substitution, we find that the positive root must be chosen to avoid an excessively large result.

We must substitute progressively for various values of b. For the bowling pin closest to the star's path, it is $b_1 = 13\,\mathrm{au}$. For the next one, it is $b_2 = 13\,\mathrm{au} + 6\,\mathrm{au} = 19\,\mathrm{au}$, where $6\,\mathrm{au}$ is half the spacing between the pins in the back row. Then for two pins, it is $b_3 = 25\,\mathrm{au}$, for another two $b_4 = 31\,\mathrm{au}$, for two more $b_5 = 37\,\mathrm{au}$, followed by one with $b_6 = 43\,\mathrm{au}$ and the last one with $b_7 = 49\,\mathrm{au}$.

Similarly, for L, the back row has $L_1 = 730\,\mathrm{au}$, and the rows in front are slightly smaller. However, we will neglect this since $L \gg b$ holds in all cases.

Depending on the index of b, we obtain values of $p_i = r_i/R_\odot$, as $p_1 \doteq 0.10$, $p_2 \doteq 0.21$, $p_3 \doteq 0.37$, $p_4 \doteq 0.57$ and $p_5 \doteq 0.81$. The only pins that God did not knock down are the two farthest from the star's path, with $p_6 \doteq 1.1$ and $p_7 \doteq 1.4$. The number of *knocked down* asteroids is thus $n = 8$.

Problem 40 ... leaking liquids 6 points

Consider a cylindrical container with the cross-section of $S = 42.0\,\mathrm{cm}^2$. Water is poured into the container to a height of $h = 5.0\,\mathrm{cm}$ above which is olive oil with a density of $\rho_{\mathrm{oil}} = 910\,\mathrm{kg{\cdot}m^{-3}}$ with the same layer thickness h, and ethanol above the oil with, also with the layer thickness h and a density of $\rho_{\mathrm{ethanol}} = 789\,\mathrm{kg{\cdot}m^{-3}}$. If a hole with an area of $s = 4.2\,\mathrm{mm}^2$ is opened at the bottom of the container, how long will it take for the water and oil to drain, leaving only the ethanol in the container?

Neglect surface tension and cohesion of the liquids. A numerical simulation is recommended.

Karel varied and complicated the problems.

We can calculate the speed of water flow using Bernoulli's equation where we neglect all terms except the kinetic one. From this relationship, we can determine the speed of flow depending on the pressure acting on the liquid as

$$v = \sqrt{\frac{2 \cdot p}{\rho}} \,.$$

The pressure p acting on the liquid after subtracting the atmospheric pressure can be calculated as

$$p = \sum \rho \cdot g \cdot h \,.$$

We know that the height of the liquid layers will change over time according to the relationship

$$\delta h = \frac{v \cdot s}{S} \cdot \delta t \,.$$

This gives us a system of differential equations that we can calculate analytically or using numerical simulation. We will show both solutions. We can divide the drainage of the layers into two separate processes and add their times together.

For water outflow, the following applies

$$\delta p = g \cdot h \cdot (\rho_{\text{oil}} + \rho_{\text{ethanol}}) + g \cdot h(t) \cdot \rho_{\text{water}} \,,$$

while for oil outflow, the following applies

$$\delta p = g \cdot h \cdot \rho_{\text{ethanol}} + g \cdot h(t) \cdot \rho_{\text{water}} \,.$$

In a numerical simulation, we substitute δp into the equation for velocity, the result of which we then substitute into the equation for the change in liquid height and thus recursively the change in height is again substituted into the equation with δp. We stop the process when the entire layer has drained and calculate the times for the drainage of the water and the oil, which we add together.

We decided to create the numerical simulation in Python and it had the following form:

```
import math

Ro_1 = 997
Ro_2 = 910
Ro_3 = 789
S = 0.0042
s = 0.0000042
h = 0.05
g = 9.81
dt = 0.0001
dh=0
t = 0

while dh<h:
       dp = g*h*(Ro_2+Ro_3)+g*Ro_1*(h-dh)
       v = math.sqrt(2*dp/Ro_1)
       dh = dh + v*s*dt/S
       t = t+dt

dh = 0

while dh<h:
       dp = g*h*Ro_3+g*Ro_2*(h-dh)
       v = math.sqrt(2*dp/Ro_2)
       dh = dh + v*s*dt/S
       t = t+dt

print(t)
```

For the analytical solution of the water layer draining, we define the constants for easier calculation

$$C_0 = g \cdot h \cdot (\rho_{\text{ethanol}} + \rho_{\text{oil}} + \rho_{\text{water}}) \,,$$

$$C_1 = g \cdot \rho_{\text{water}} \,,$$

which we use to rewrite the relationship with δp into the form of

$$\delta p = C_0 - C_1 \cdot \delta h(t)\,.$$

The differential equation is therefore separable and takes the form

$$\frac{\mathrm{d}h}{\mathrm{d}t} = K \cdot \sqrt{C_0 - C_1 \cdot h}\,,$$

where $K = (s/S) \cdot \sqrt{2/\rho_{\text{water}}}$.

By solving this equation for $\delta h(t)$, we obtain the equation

$$\delta h(t) = \frac{C_0 - (\sqrt{C_0} - \frac{C_1 \cdot K}{2} \cdot t)^2}{C_1}\,.$$

By solving it, we obtain the final relationship for the drainage of the water layer for time t

$$t_1 = \frac{2}{C_1 \cdot K} \cdot (\sqrt{C_0} - \sqrt{C_0 - C_1 \cdot h})\,.$$

By slightly adjusting the constants, we can calculate the time for the drainage of the oil layer and we add these two times together

$$\begin{aligned}
C_2 &= g \cdot h \cdot (\rho_{\text{ethanol}} + \rho_{\text{oil}})\,,\\
C_3 &= g \cdot \rho_{\text{oil}}\,,\\
K_1 &= \frac{s}{S} \cdot \sqrt{\frac{2}{\rho_{\text{oil}}}}\,,\\
t_2 &= \frac{2}{C_3 \cdot K_1} \cdot (\sqrt{C_2} - \sqrt{C_2 - C_3 \cdot h})\,,\\
t &= t_1 + t_2 = 78s\,.
\end{aligned}$$

Problem 41 ... the cosmos getting far, far away — 7 points

How far away would stars and galaxies have to be – objects that would otherwise be relatively at rest with respect to us, but which we observe with such a redshift due to the expansion of the universe that the wavelengths of their radiation are twice as large as at the moment they were emitted? We are asking about the "present-day" distance, not the distance they had at the time the signal was emitted. Assume that the universe is homogeneous and isotropic, that Einstein's equations and the standard ΛCDM cosmological model hold, and that the universe expands with a Hubble constant that is constant in time, with value $H_0 = 68\,\mathrm{km{\cdot}s^{-1}{\cdot}Mpc^{-1}}$.

Karel was thinking about a very distant zone.

In the considered ΛCDM cosmological model, the Friedmann equations hold, according to which the Hubble parameter is given by

$$H(t) = H_0 \sqrt{\Omega_{\mathrm{r},0} a^{-4} + \Omega_{\mathrm{m},0} a^{-3} + \Omega_{k,0} a^{-2} + \Omega_{\Lambda,0}}\,, \tag{8}$$

where $a \equiv a(t)$ is the scale factor describing the expansion of the universe, and the dimensionless quantities $\Omega_{x,0}$ describe the present-day fractions of radiation ($\Omega_{r,0}$), matter ($\Omega_{m,0}$), curvature ($\Omega_{k,0}$), and dark energy ($\Omega_{\Lambda,0}$). From the assumption that the Hubble "constant" is constant in time, it follows that all terms that vary with time must be zero. Therefore, the cosmological model under consideration must be a flat universe ($\Omega_{k,0} = 0$) containing only dark energy ($\Omega_{r,0} = 0, \Omega_{m,0} = 0, \Omega_{\Lambda,0} = 1$).

Solution via the scale factor

Because of the expansion of the universe, physical distances increase, which can be expressed using the scale factor $a(t)$ as

$$d(t) = a(t)\chi, \tag{9}$$

where $d(t)$ is the physical distance from the observer and χ is a dimensionless constant called the comoving distance, which is related to the physical distance at time t_0. The scale factor at the present time (t_0) is conventionally $a(t_0) = 1\,\mathrm{m}$. The expansion of the universe also stretches wavelengths, which is described using the redshift z, for which, by definition

$$z = \frac{\lambda_{\mathrm{obs}} - \lambda_{\mathrm{e}}}{\lambda_{\mathrm{e}}} = \frac{2\lambda_{\mathrm{e}} - \lambda_{\mathrm{e}}}{\lambda_{\mathrm{e}}} = 1,$$

where λ_e is the emitted wavelength and $\lambda_{obs} = 2\lambda_e$ is the observed wavelength. The effect of expansion on the wavelength can be derived from (9):

$$\frac{\lambda_{\mathrm{e}}}{a(t_{\mathrm{e}})} = \frac{\lambda_{\mathrm{obs}}}{a(t_0)}$$

$$a(t_{\mathrm{e}}) = \frac{\lambda_{\mathrm{e}}}{\lambda_{\mathrm{obs}}} a(t_0) = \frac{a(t_0)}{1+z}\,.$$

The time dependence of the scale factor $a(t)$ is obtained from the definition of the Hubble parameter $H(t) \equiv \frac{\dot{a}}{a}$, where in this problem we assume $H(t) = H_0$, yielding

$$H_0 = \frac{\frac{da}{dt}}{a} \to \frac{da}{a} = H_0 dt$$

$$\int_{a(t_0)}^{a(t)} \frac{da}{a} = H_0 \int_{t_0}^{t} dt$$

$$a(t) = a(t_0) e^{H_0(t-t_0)}\,.$$

According to the FLRW metric for our cosmological model, the comoving distance is given by

$$\chi = \int_{t_{\mathrm{e}}}^{t_0} \frac{c\,dt}{a(t)}, \tag{10}$$

from which, after substitution and simplification, we obtain

$$\chi = \frac{c}{a(t_0)} \int_{t_{\mathrm{e}}}^{t_0} e^{-H_0(t-t_0)} dt = \frac{c}{a(t_0)H_0}\left(e^{-H_0(t_{\mathrm{e}}-t_0)} - 1\right)$$

$$= \frac{c}{a(t_0)H_0}\left(\frac{a(t_0)}{a(t_{\mathrm{e}})} - 1\right) = \frac{cz}{a(t_0)H_0}.$$

Thus, the present-day physical distance is:

$$d(t_0) = a(t_0)\chi = \frac{cz}{H_0} = \frac{c}{H_0} \doteq 4.4 \cdot 10^3 \,\mathrm{Mpc}.$$

Solution via the redshift

From (8), for a flat universe containing only dark energy, we have $H(z) = H_0$. The integral in (10) can equivalently be rewritten as

$$\chi = \frac{c}{a(t_0)} \int_0^z \frac{dz}{H(z)} = \frac{c}{a(t_0)H_0} \int_0^z dz = \frac{cz}{a(t_0)H_0},$$

Yielding the same result without needing to compute the time dependence of the scale factor.

Problem 42 ... underwater light 7 points

Consider a point source of unpolarized light radiating in all directions. The source is submerged at a depth $h = 1.00\,\mathrm{m}$ *below a (sufficiently large) calm water surface; the bottom beneath it is perfectly dark. What percentage of the radiant power is transmitted above the surface? The refractive index of water is* $n_i = 1,33$*; the refractive index of air is* $n_t = 1.00$*. Assume that the water absorbs none of the light. Approximate the intensity transmission coefficient by its value at normal incidence everywhere where light is transmitted.*

Petr was reminiscing about the Optics course.

To solve the problem we need the Fresnel transmission coefficients

$$t_{\parallel} = \frac{2n_i \cos\theta_i}{n_t \cos\theta_i + n_i \cos\theta_t}\,,$$
$$t_{\perp} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}\,,$$

where n denotes the refractive index of the medium and θ is the angle from the normal to the interface at which the light propagates; the lower indices indicate whether the quantity belongs to the medium from which is the light incoming (i) or to the medium into which it transmits (t). We will also need the coefficients in intensity form

$$T_{\parallel} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} |t_{\parallel}|^2\,,$$
$$T_{\perp} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} |t_{\perp}|^2\,.$$

The transmitted intensity is then

$$I_t = T_{\parallel} I_{\parallel} + T_{\perp} I_{\perp}\,.$$

If, however, the light is incident on the interface at normal incidence (i.e. $\theta_i = \theta_t = 0$), both coefficients coincide and reduce to a single coefficient

$$T = \frac{4n_i n_t}{(n_i + n_t)^2}$$

and one simply has

$$I_t = TI\,.$$

We approximate the transmitted intensity by this relation everywhere on the interface where light is transmitted. The intensity is not uniform under the interface—intensity is the radiant power per unit area, and because power (energy) is conserved, the intensity, i.e. the magnitude of the Poynting vector[3] $|\vec{\mathcal{S}}|$ at a distance r from the source satisfies

$$I = |\vec{\mathcal{S}}| = \frac{P}{4\pi r^2}\,.$$

However, since this is a vector magnitude, when asking for the power transmitted through a plane we must consider only the component of the Poynting vector normal to the interface $\mathcal{S}_\perp$. Thus

$$\mathcal{S}_\perp = \frac{P}{4\pi r^2}\cos\vartheta\,,$$

where ϑ is the angle between the Poynting vector and the plane normal.

Combining the results above, for the differential transmitted power $\mathrm{d}P_t$ we write

$$\mathrm{d}P_t = \frac{n_i n_t}{(n_i+n_t)^2}\frac{P}{\pi r^2}\cos\vartheta\,\mathrm{d}S\,.$$

W parametrise the surface in polar coordinates with the origin at the intersection of the surface and the axis through the source. For the distance from the source r, and for the cosine of the angle between the Poynting vector and the normal to the plane $\cos\vartheta$, we have

$$r = \sqrt{\rho^2+h^2}\,,$$

$$\cos\vartheta = \frac{h}{\sqrt{\rho^2+h^2}}\,,$$

where ρ is the radial coordinate. Upon transforming to polar coordinates we must multiply by the Jacobian, which equals ρ. Note also that the integrand does not depend on the azimuthal coordinate φ, hence we may perform that integration immediately by multiplying by 2π. After simplification we obtain

$$\mathrm{d}P_t(\rho) = \frac{2n_i n_t P}{(n_i+n_t)^2}\frac{\rho h}{(\rho^2+h^2)^{3/2}}\,\mathrm{d}\rho\,.$$

We now integrate the transmitted power. We observe that for some maximum radius R the critical angle is reached, and for any $\rho > R$ the transmitted power is zero.

Integrating $\mathrm{d}P_t$ gives

$$P_t = \frac{2n_i n_t Ph}{(n_i+n_t)^2}\int_0^R \frac{1}{(\rho^2+h^2)^{3/2}}\rho\,\mathrm{d}\rho \implies P_t = \frac{2n_i n_t P}{(n_i+n_t)^2}\left(1-\frac{h}{\sqrt{R^2+h^2}}\right)\,.$$

A careful reader will note that the second term in the parentheses is simply the cosine of the critical angle θ_m. For the critical angle one has

$$\sin\theta_m = \frac{n_t}{n_i} \implies \cos\theta_m = \sqrt{1-\frac{n_t^2}{n_i^2}}\,.$$

[3] For completeness, the Poynting vector represents the magnitude and direction of the energy flux carried by the electromagnetic field (of which light is an instance). One has $\vec{\mathcal{S}} = I\vec{e}$, where I is the intensity and $\vec{e}$ is the propagation direction of the wave.

Substituting into the expression for P_t yields

$$P_t = \frac{2n_t P}{(n_i + n_t)^2} \left(n_i - \sqrt{n_i^2 - n_t^2} \right) .$$

This gives a very good approximate result

$$\frac{P_t}{P} \doteq 16.7\,\% \,,$$

which differs from the exact calculation by only about 0.9%.

Remark on the derivation of intensity transmission and reflection coefficients

We now briefly indicate how the forms of the transmission and reflection coefficients used above may be obtained. The classical coefficients for the electric field amplitude E are derived from the continuity conditions for the electric and magnetic fields at the interface between two media, as provided by electromagnetic theory. We will not present the detailed derivation; instead we assume the forms of $t_{\|}$ and $t_{\perp}$ and derive $T_{\|}$ and $T_{\perp}$ from them. Consider the Poynting vector $\vec{\mathcal{S}}$ below the interface and $\vec{\mathcal{S}}'$ above the interface. The incident power on an area element $\mathrm{d}S$ is

$$P = \vec{\mathcal{S}} \cdot \mathrm{d}\vec{S} \,,$$
$$P' = \vec{\mathcal{S}}' \cdot \mathrm{d}\vec{S} \,,$$

where $\mathrm{d}\vec{S}$ is the area element $\mathrm{d}S$ multiplied by its normal $\vec{n}$. P is the incident power and P' is the transmitted power. Define the transmission ratio T as

$$T \equiv \frac{P'}{P} = \frac{\vec{\mathcal{S}}' \cdot \mathrm{d}\vec{S}}{\mathcal{S} \cdot \mathrm{d}\vec{S}} = \frac{|\vec{\mathcal{S}}'||\,\mathrm{d}\vec{S}|\cos\theta_t}{|\vec{\mathcal{S}}||\,\mathrm{d}\vec{S}|\cos\theta_i} = \frac{|\vec{\mathcal{S}}'|\cos\theta_t}{|\vec{\mathcal{S}}|\cos\theta_i} = \frac{I_t \cos\theta_t}{I\cos\theta_i} \,.$$

For the intensity of a monochromatic plane wave in a medium with refractive index n_j one has

$$I_j \propto n_j |\vec{E}|^2 \,,$$

where $\vec{E}$ is the electric field vector. If under the interface $\vec{E} = \vec{E}_{\|} + \vec{E}_{\perp}$, then after transmission above the interface

$$\vec{E}^t_{\|} = t_{\|}\vec{E}_{\|} \,,$$
$$\vec{E}^t_{\perp} = t_{\|}\vec{E}_{\|} \,.$$

Substituting into the expression for T yields

$$T = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} \frac{|t_{\|}\vec{E}_{\|} + t_{\perp}\vec{E}_{\perp}|^2}{|\vec{E}_{\|} + \vec{E}_{\perp}|^2} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} \frac{|t_{\|}|^2|\vec{E}_{\|}|^2 + |t_{\perp}|^2|\vec{E}_{\perp}|^2}{|\vec{E}_{\|}|^2 + |\vec{E}_{\perp}|^2} \,.$$

The equalities $|\vec{E}_{\|} + \vec{E}_{\perp}|^2 = |\vec{E}_{\|}|^2 + |\vec{E}_{\perp}|^2$ and $|t_{\|}\vec{E}_{\|} + t_{\|}\vec{E}_{\perp}|^2 = |t_{\|}|^2|\vec{E}_{\|}|^2 + |t_{\perp}|^2|\vec{E}_{\perp}|^2$ follow from the Pythagorean theorem because the summed vectors are orthogonal.

Since we typically work with the separate intensity components associated with $\vec{E}_\parallel$ and $\vec{E}_\perp$, we introduce coefficients $T_\parallel$ and $T_\perp$ such that

$$I_t = TI = T_\parallel I_\parallel + T_\perp I_\perp \,.$$

This yields exactly

$$T_\parallel = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} |t_\parallel|^2 \,,$$
$$T_\perp = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} |t_\perp|^2 \,.$$

Problem 43 ... hmm, where did i put that? 8 points

You've surely noticed the increase in entropy in your room, or that things tend to diffuse (especially when your roommate is away). Recently, Lego stored his favorite Brownian particle in the minimum of a 2D potential shaped as $U(x,y) = C(x^2+y^2)$, where $C = 1.0\cdot 10^{-20}\ \mathrm{J\cdot m^{-2}}$. After a very long time, he went to look for it. What is the probability that he finds it within a distance less than $d = 50\,\mathrm{cm}$ from the spot where he left it? This task was conceived during the summer, so the temperature in the dorm room is $T = 40\,{}^\circ\mathrm{C}$.

As a theoretical physicist, Lego loves physics that have something to do with reality.

Since a very long time passed between dropping and searching for the particle, its position will be determined by the Boltzmann distribution given by the potential and the temperature. The probability that the particle is at point (x,y) is proportional to $\exp(-U(x,y)/(k_\mathrm{B}T))$.

The normalization constant is commonly called the partition function, denoted by Z. We obtain it from the condition that the integral of the probability density over all space must equal 1

$$Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(\frac{-C(x^2+y^2)}{k_\mathrm{B}T}\right) \mathrm{d}x\,\mathrm{d}y \,.$$

For simplicity, we switch to polar coordinates; in other words, we introduce the substitution $x = r\cos\varphi, y = r\sin\varphi$, which also means that instead of rectangles with sides $\mathrm{d}x, \mathrm{d}y$ we move to rectangles with sides $r\,\mathrm{d}\varphi, \mathrm{d}r$. We insert this substitution and integrate using another substitution, where we substitute the exponent as u

$$Z = \int_0^\infty \int_0^{2\pi} \exp\left(\frac{-Cr^2}{k_\mathrm{B}T}\right) r\,\mathrm{d}\varphi\,\mathrm{d}r = \pi\frac{k_\mathrm{B}T}{C}\int_0^\infty \exp(-u)\,\mathrm{d}u = \pi\frac{k_\mathrm{B}T}{C}\left[-e^{-u}\right]_0^\infty = \pi\frac{k_\mathrm{B}T}{C} \,.$$

In a similar way, we integrate the region for which $r < d$, only now we must be careful about the change of the upper limit when substituting to u

$$P(r<d) = \int_0^d \int_0^{2\pi} p(r,\varphi) r\,\mathrm{d}\varphi\,\mathrm{d}r = 2\pi\int_0^d \frac{1}{Z}\exp\left(\frac{-Cr^2}{k_\mathrm{B}T}\right) r\,\mathrm{d}r$$
$$= \left[-e^{-u}\right]_0^{\frac{Cd^2}{k_\mathrm{B}T}} = 1 - e^{-\frac{Cd^2}{k_\mathrm{B}T}} = 0.44 \,.$$

Problem 44 ... racing against the rain 6 points

David and Terka enjoy playing catch with a ball. During the summer, they were able to throw it a maximum distance of $5.0\,\mathrm{m}$*. The ball has a radius of* $r = 15\,\mathrm{cm}$ *and a mass of* $m = 300\,\mathrm{g}$*. By how many centimeters (in absolute value) will the distance they can throw the ball differ in winter? The air density outside the ball is in summer* $\rho_1 = 1.12\,\mathrm{kg\cdot m^{-3}}$ *and* $\rho_2 = 1.29\,\mathrm{kg\cdot m^{-3}}$ *in winter. The atmosphere does not affect the athletic abilities of David or Terka, nor the physical properties of the ball.*

David was thinking who will play catch with him when it rains.

At the speeds needed to throw the ball to 5 meters, the air around it will certainly no longer flow laminarily. For this reason, we use Newton's drag law for the resistance of the medium

$$F_D = \frac{1}{2}\rho v^2 C_D S\,,$$

where ρ is the air density, v is the ball's speed, C_D is the drag coefficient, whose value for a sphere is $C_D = 0.47$, and S is the cross–sectional area, i.e., $S = \pi r^2$. The force acts opposite to the direction of motion.

We need to decompose the motion into x and y directions; therefore, we also need to decompose the force into these directions. The force is proportional to v^2, so its x-component is proportional to vv_x and the y-component is proportional to vv_y. From the Pythagorean theorem, it follows that the decomposition preserves the magnitude of the force, and it is intuitive that it preserves the direction as well. The differential equations for the velocity components will then be

$$\begin{aligned}\dot{v}_x &= -\frac{1}{2m}\rho C_D S v v_x\\ \dot{v}_y &= -\frac{1}{2m}\rho C_D S v v_y - g + \frac{V\rho g}{m}\,,\end{aligned}$$

where the last term is due to the buoyant force, and $V = 4/3\pi r^3$ is the volume of the ball. This force is negligible compared to gravity, but not negligible compared to drag. The recommended procedure is to solve these differential equations numerically.

Specifically, we first use a binary search to find the initial throwing speed, then we measure both distances and subtract them. This code assumes that the ball is thrown at an angle of 45 degrees. We also created code that iterates through different angles around that value and selects the longest distance – the result differed only by one millimeter.

```
dt=0.0001
g=9.81
m=0.3
S=pi*0.15^2
V=4/3*pi*0.15^3
Cd=0.47
rho=1.12

v_below=0
v_above=10

for i=1:20 # binary search for the initial velocity for which the ball will fly 5m

        vx=(v_below+v_above)/2
        vy=vx
```

```
        x=0
        y=0



        while y>=0 # The ball flies as long as it is above the ground
                v=sqrt(vx*vx+vy*vy) # current speed
                vy-=g*dt + 1/2*rho*vy*v*S*Cd*dt/m - V*rho*g*dt/m # change of v_y
                vx-= 1/2*rho*vx*v*S*Cd*dt/m # and of v_x

                x+=vx*dt # movement
                y+=vy*dt
        end

        if x>5
                v_above=(v_below+v_above)/2
        else
                v_below=(v_below+v_above)/2
        end
end

rho=1.29

vx=(v_below+v_above)/2 # we set the initial speed as the average of the interval
vy=(v_below+v_above)/2 # which we got by binary search
x=0
y=0

while y>=0
        v=sqrt(vx*vx+vy*vy)
        vy-=g*dt + 1/2*rho*vy*v*S*Cd*dt/m - V*rho*g*dt/m
        vx-= 1/2*rho*vx*v*S*Cd*dt/m


        x+=vx*dt
        y+=vy*dt
end
x1=x

rho=1.12 # and with the same speed we throw it once more in the original density,
# because it will not reach exactly 5m

vx=(v_below+v_above)/2
vy=(v_below+v_above)/2
x=0
y=0

while y>=0
        v=sqrt(vx*vx+vy*vy)
        vy-=g*dt + 1/2*rho*vy*v*S*Cd*dt/m - V*rho*g*dt/m
        vx-= 1/2*rho*vx*v*S*Cd*dt/m


        x+=vx*dt
        y+=vy*dt
end
x2=x
x2-x1
```

The code returns 0.128, meaning they must move 13 cm closer.

Problem 45 ... blowing up a balloon reloaded

7 points

When inflating a balloon, we can, from a certain radius onward, assume with good accuracy that the elastic energy increases directly proportionally to the growing surface area of the balloon. In this dimension range, Jirka found his previously inflated balloon, which in the meantime had deflated to a radius of 5.0 cm *(the balloon is approximately spherical). Since the balloon is rather small, he decided to inflate it back to its original size. Balloon inflation is, however, physically demanding. Jirka wants to know how much work his muscles perform when he inflates the balloon using a maximal possible exhalation.*

Assume that after inhaling, Jirka has 5.5 l *of air in his lungs, and after maximal exhalation,* 1.5 l *of air remain in his lungs. Further assume that the pressure and temperature of the air in his lungs is at all times equal to those in the balloon, and approximate the inflation as an isothermal process (thus we consider the lungs and the balloon as a single connected vessel undergoing a quasistatic process). The air temperature in the balloon is* 25 °C *and the initial pressure* 107 kPa*. This pressure value corresponds both to the situation before the inflation begins (i.e., for the isolated balloon) and at its start. The pressure in Jirka's lungs adjusts to match the initial pressure in the balloon. Compute the work of Jirka's lungs only from the moment the inflation starts. Assume that the work is determined solely by the change in lung volume and the difference between the gas pressure and atmospheric pressure (the surrounding atmospheric pressure assists the lungs during gas compression).*

Jirka ressurected a problem from two years ago.

The ideal gas equation of state holds

$$pV = nRT\,,$$

which we consider for the entire "container", i.e. the balloon connected to Jirka's lungs with total volume

$$V = V_1 + \frac{4}{3}\pi r^3\,,$$

where V_1 is the volume of the lungs and r is the radius of the balloon. During inflation we assume an isothermal process, so the right-hand side of the equation is constant. The work is computed according to the instructions in the statement as

$$\mathrm{d}W = -(p - p_\mathrm{a})\,\mathrm{d}V_1\,.$$

Note that this is not an unphysical approach. The muscles are assisted by the surrounding atmospheric pressure when contracting. If we imagine the equilibrium process during which the balloon inflates as lungs first reduce their volume V_1 (thereby performing work $-(p - p_\mathrm{a})\,\mathrm{d}V_1$) and then the balloon being inflated, the work during balloon inflation is performed by the gas and the temperature remains equalized.

To obtain the work, we must integrate the previous relation. We do not yet know the exact relation between p and V_1. One way to proceed is to note that $V = V_1 + (4/3)\pi r^3$, hence

$$\mathrm{d}V_1 = \mathrm{d}V - 4\pi r^2\,\mathrm{d}r\,.$$

Substituting yields

$$\mathrm{d}W = -p\,\mathrm{d}V + 4\pi r^2 p\,\mathrm{d}r + p_\mathrm{a}\,\mathrm{d}V_1\,.$$

The first term gives the work for the isothermal process, the second term can be rewritten as a function $p(r)$ and the third term can be integrated directly as

$$\int p_\mathrm{a}\,\mathrm{d}V_1 = p_\mathrm{a}(V_{1f} - V_{1i}) = -p_\mathrm{a}\Delta V_1\,,$$

where $\Delta V_1 = 4\,\mathrm{l}$ is the change of lung volume, the subscript i denotes the initial state and f the final state.

The work in the isothermal process is given by

$$W = nRT\ln\frac{p_f}{p_i} = -nRT\ln\frac{V_f}{V_i}\,.$$

In this equation the factor nRT is computed from the initial condition, since the initial pressure and volume are known

$$nRT = p_iV_i \doteq 645\,\mathrm{J}\,.$$

To complete the computation of the work we must determine the final state. For this we use the equation of state. However it contains two unknown variables p and V (we know the final lung volume but not the balloon radius). We must therefore add an additional equation. This will be the dependence of pressure on the radius, which we derive from the prescribed elastic behavior of the balloon.

We know that increasing the surface area of the balloon by $\mathrm{d}S$[4] requires performing work $\mathrm{d}W_2$ against forces in the balloon, hence

$$\mathrm{d}W_2 = \sigma\,\mathrm{d}S = 8\pi\sigma r^2\,\mathrm{d}r\,,$$

the supplied energy must be proportional to the change in area, as stated in the problem.

The same work can be expressed using pressure. The balloon is spherical, so the resultant forces act toward the center (the balloon tends to contract). When the balloon is increased by a small radius $\mathrm{d}r$ the work performed is

$$\mathrm{d}W_2 = F\cdot\mathrm{d}r = p_2\cdot S\cdot\mathrm{d}r = p_2\cdot 4\pi r^2\,\mathrm{d}r\,,$$

where p_2 is the pressure produced by the balloon at radius r. Because $\mathrm{d}r$ is small, upon inflation by Δr, the force F and the area S change negligibly; i.e. terms of order $\mathcal{O}(\mathrm{d}r^2)$ may be neglected.

Comparing the two equations we obtain

$$p_2\cdot 4\pi r^2\,\mathrm{d}r = \sigma\cdot 8\pi r^2\,\mathrm{d}r\,,$$

from which the dependence of pressure on the radius follows

$$p_2 = \frac{2\sigma}{r}\,.$$

One must realize that the previous equation does not yet express the air pressure inside the balloon. The air is also subject to atmospheric pressure $p_\mathrm{a} = 101\,325\,\mathrm{Pa}$. The total air pressure is then $p_2 + p_\mathrm{a}$.

[4] The area $\mathrm{d}S$ is infinitesimal, although the relation for the work would also hold for a finite ΔS.

From the problem statement we know the initial pressure p_i and radius r_i. From these we determine the constant 2σ as $2\sigma = (p_i - p_\mathrm{a})r_i$. We can now integrate the remaining term in the equation for dW

$$\int 4\pi r^2 \left(p_\mathrm{a} + (p_i - p_\mathrm{a})\frac{r_i}{r}\right) \mathrm{d}r = \frac{4}{3}\pi(r_f^3 - r_i^3)p_\mathrm{a} + 2\pi r_i(r_f^2 - r_i^2)(p_i - p_\mathrm{a})\,,$$

thereby expressing the work as

$$W = -nRT \ln \frac{V_f}{V_i} + \frac{4}{3}\pi(r_f^3 - r_i^3)p_\mathrm{a} + 2\pi r_i(r_f^2 - r_i^2)(p_i - p_\mathrm{a}) - p_\mathrm{a}\Delta V_1\,.$$

It remains to determine the final radius of the balloon. We use the equation of state for the final state

$$\left(p_\mathrm{a} + (p_i - p_\mathrm{a})\frac{r_i}{r_f}\right)\left(V_{1f} + \frac{4}{3}\pi r_f^3\right) = nRT\,,$$

where $V_{1f} = 1.5\,\mathrm{l}$ is the final lung volume. This equation must now be solved numerically for the radius r_f, which yields two solutions. We are interested only in the one larger than the initial radius, namely $r_f = 10.4\,\mathrm{cm}$. Note that the elastic effect of the balloon is actually rather small; neglecting the key term in the equation would give $10.5\,\mathrm{cm}$. Substituting into the expression for the work yields the result $W \doteq 14\,\mathrm{J}$.

Problem 46 ... a spring and a magnet 8 points

Consider a spring of stiffness $k = 50.0\,\mathrm{mN{\cdot}cm^{-1}}$, hanging vertically, on which a magnet of mass $M = 110\,\mathrm{g}$ and magnetic moment $m_1 = 2.00\,\mathrm{A{\cdot}m^2}$ is attached. The magnet also acts as a weight. On the floor below the magnetic weight is placed another magnet with the magnetic moment $m_2 = 10.0\,\mathrm{A{\cdot}m^2}$. Calculate the period of the small oscillations, assuming that the magnets are of negligible size, and thus can be considered as magnetic dipoles, and are moving slowly enough that we can neglect any losses. The magnets are oriented in such a direction that they repel each other, and the magnet acting as a weight is exactly such that the deflection of the spring does not change when it is suspended.

Magnets are attracted to each other and their relationship can even be harmonic.

We begin by setting the problem in a coordinate system. Let x denote the vertical coordinate, with the origin positioned at the equilibrium point of the weight. The orientation of x is chosen such that the positive part of the axis points downward, making the gravitational force positive. Then, h_0 represents the initial position of the weight above the braking magnet. For the magnetic force between two magnets as a function of displacement x, we have

$$F_\mathrm{m}(x) = \frac{3\mu_0}{2\pi}\frac{m_1 m_2}{(h_0 - x)^4}. \tag{11}$$

Next, we construct the equation of motion. The net force is $M\ddot{x}$, which has the same sign as the gravitational force when the weight is pulled downward. This gives

$$M\ddot{x} = Mg - F_\mathrm{m}(x) - kx \quad\Rightarrow\quad M\ddot{x} + F_\mathrm{m} + kx = Mg. \tag{12}$$

Before forming the harmonic oscillator equation, we calculate the initial position h_0. We do this by setting $M\ddot{x} = 0$ and $x = 0$ in the equation (12), then substituting the expression for the magnetic force (11). This yields

$$0 = Mg - F_\mathrm{m}(0) \quad \Rightarrow \quad Mg = \frac{3\mu_0}{2\pi}\frac{m_1 m_2}{h_0^4} \quad \Rightarrow \quad h_0 = \sqrt[4]{\frac{3\mu_0 m_1 m_2}{2\pi Mg}} \tag{13}$$

Now we express the equation of the harmonic oscillator. Substituting the expression for F_m from the equation (11) into the equation (12) and dividing by the mass, we obtain

$$\ddot{x} + \frac{3\mu_0}{2\pi M}\frac{m_1 m_2}{(h_0 - x)^4} + \frac{k}{M}x = g \tag{14}$$

Since the origin of the coordinate system is at the equilibrium point and the oscillations are small, we have $|x| \ll h_0$. To transform the equation (14) into the form of the harmonic oscillator equation, we approximate the term $(h_0 - x)^{-4}$ using a Taylor polynomial. Let us define $f(x) := = (h_0 - x)^{-4}$. Since 0 belongs to the domain of $f(x)$, we expand the Taylor polynomial around 0

$$f(x) = \frac{1}{(h_0 - x)^4} \approx T_{0,n}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k \approx \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x.$$

Due to the approximation, all terms of the polynomial with exponents greater than 2 are negligible. Thus, we only compute $f'(0)$

$$\frac{1}{(h_0 - x)^4} \approx \frac{1}{h_0^4} + \left(-4(h_0 - x)^{-5}(-1)\right)\Big|_{x=0} x = \frac{1}{h_0^4} + \frac{4x}{h_0^5}.$$

We substitute this into the equation (14)

$$\ddot{x} + \frac{3\mu_0 m_1 m_2}{2\pi M}\left(\frac{1}{h_0^4} + \frac{4x}{h_0^5}\right) + \frac{k}{M}x = g \quad \Rightarrow$$

$$\ddot{x} + \left(\frac{6\mu_0 m_1 m_2}{\pi M h_0^5} + \frac{k}{M}\right)x = g - \frac{3\mu_0 m_1 m_2}{2\pi M h_0^4}.$$

Finally, by substituting h_0 from the equation (13), we obtain

$$\ddot{x} + \left(4g\sqrt[4]{\frac{2\pi Mg}{3\mu_0 m_1 m_2}} + \frac{k}{M}\right)x = 0. \tag{15}$$

Comparing the equation (15) to the equation of the harmonic oscillator $\ddot{x} + \omega^2 x = 0$, we find

$$\omega^2 = \frac{4\pi^2}{T^2} = \left(4g\sqrt[4]{\frac{2\pi Mg}{3\mu_0 m_1 m_2}} + \frac{k}{M}\right) \quad \Rightarrow \quad T = \frac{2\pi}{\sqrt{4g\sqrt[4]{\frac{2\pi Mg}{3\mu_0 m_1 m_2}} + \frac{k}{M}}}. \tag{16}$$

Substituting into equation (16), we get

$$T \doteq 0.233\,\mathrm{s}\,.$$

Problem 47 ... underwater light reloaded 9 points

Consider a point source of unpolarized light radiating in all directions. The source is submerged at a depth of $h = 1.00\,\mathrm{m}$ below a (sufficiently large) calm water surface. What percentage of the radiant power is transmitted above the surface? The refractive index of water is $n_i = 1.33$; the refractive index of air is $n_t = 1.00$. Assume that water absorbs none of the light and use the exact values of the transmission coefficients.

Petr liked the graph.

As in the approximate version of the problem, we will need the Fresnel coefficients in both amplitude and intensity form,

$$t_\parallel = \frac{2n_i \cos\theta_i}{n_t \cos\theta_i + n_i \cos\theta_t}\,,$$
$$t_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}\,,$$
$$T_\parallel = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\left|t_\parallel\right|^2\,,$$
$$T_\perp = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\left|t_\perp\right|^2\,.$$

Because we are working with an unpolarized source for which $I_\parallel = I_\perp = \frac{1}{2}I$, we can write

$$I_t = \frac{1}{2}\left(T_\parallel + T_\perp\right) I\,.$$

Intensity is the radiant power per unit area. Since power (due to energy conservation) is conserved, the intensity (i.e., the magnitude of the Poynting vector $|\vec{\mathcal{S}}|$) at a distance r from the source satisfies

$$I = |\vec{\mathcal{S}}| = \frac{P}{4\pi r^2}\,.$$

For the passage of light through a plane, we must also consider only the component of the Poynting vector perpendicular to it, namely

$$\mathcal{S}_\perp = \frac{P}{4\pi r^2}\cos\vartheta\,,$$

where ϑ is the angle between the Poynting vector and the normal to the plane.

Combining all results above, we may write for the differential transmitted power $\mathrm{d}P_t$

$$\mathrm{d}P_t = \frac{P\cos\vartheta}{2\pi r^2} n_i n_t \cos\theta_t \cos\theta_i \left(\frac{1}{\left(n_t \cos\theta_i + n_i \cos\theta_t\right)^2} + \frac{1}{\left(n_i \cos\theta_i + n_t \cos\theta_t\right)^2}\right)\mathrm{d}S$$

We parametrize the surface by polar coordinates with the origin at the intersection of the surface and the axis passing through the source. Then

$$\cos\vartheta = \frac{h}{\sqrt{\rho^2 + h^2}}\,,$$

where ρ is the radial coordinate given by the distance from the axis. For the cosine of the angle of incidence,

$$\cos\theta_i = \frac{h}{\sqrt{\rho^2 + h^2}}\,.$$

From Snell's law ($n_i \sin\theta_i = n_t \sin\theta_t$) we have for the cosine of the transmission angle

$$\cos\theta_t = \sqrt{1 - \frac{n_i^2}{n_t^2}\frac{\rho^2}{\rho^2 + h^2}}\,.$$

We can now substitute everything into the expression for $\mathrm{d}P_t$. After multiplying by the Jacobian of polar coordinates ρ and integrating over φ (i.e., multiplying by 2π), we obtain

$$\mathrm{d}P_t(\rho) = \frac{P n_i n_t}{(\rho^2 + h^2)^{3/2}} h^2 \rho \sqrt{h^2 + \rho^2\left(1 - \frac{n_i^2}{n_t^2}\right)} \left(\frac{1}{\left(n_t h + n_i\sqrt{h^2 + \rho^2\left(1 - \frac{n_i^2}{n_t^2}\right)}\right)^2} + \right.$$
$$\left. + \frac{1}{\left(n_i h + n_t\sqrt{h^2 + \rho^2\left(1 - \frac{n_i^2}{n_t^2}\right)}\right)^2} \right) \mathrm{d}\rho\,.$$

We now wish to integrate the transmitted power. Since analytic evaluation is not feasible, we choose numerical integration. As before, there is no need to integrate to arbitrarily large ρ; we integrate only up to $\rho_{\max} = R$, where the critical angle is reached. As in the approximate version of the problem, we can express it using the knowledge of the critical angle as

$$R = \frac{h n_t}{\sqrt{n_i^2 - n_t^2}}\,,$$
$$\rho \in \left\langle 0, \frac{h n_t}{\sqrt{n_i^2 - n_t^2}} \right\rangle\,.$$

We then integrate the expression for P_t in Python using the following code:

```
N = 100_000

def sqrt(x):
       return x ** 0.5

def support_function(h, ni, nt, rho):
       return( sqrt( h**2 + rho**2*(1-(ni/nt)**2) ) )

def dPt(h, ni, nt, P, rho):
return( (P*ni*nt)/(rho**2+h**2)**(3/2)*h**2*rho*
       support_function(h, ni, nt, rho)*( 1/(nt*h+ni*
       support_function(h, ni, nt, rho))**2 +
       1/(ni*h+nt*support_function(h, ni, nt, rho))**2 ) )

rho_min = 0
rho_max = (h*nt)/sqrt(ni**2 - nt**2)

def integrate(h, ni, nt, P, rho_min, rho_max, n):
       delta_rho = (rho_max - rho_min) / n
       total = 0
       for i in range(n):
               rho = rho_min + (i + 0.5) * delta_rho
```

```
                total += dPt(h, ni, nt, P, rho) * delta_rho
        return total

result = integrate(h, ni, nt, P, rho_min, rho_max, N)
print(result)
```

This yields the total result

$$\frac{P_t}{P} \doteq 15.8\% \,.$$

Note that P_t does not depend on the depth h at which the source is submerged. One can verify this by observing that the parameter h can be eliminated from the integral for P_t by the substitution $\rho \to \rho/h$.

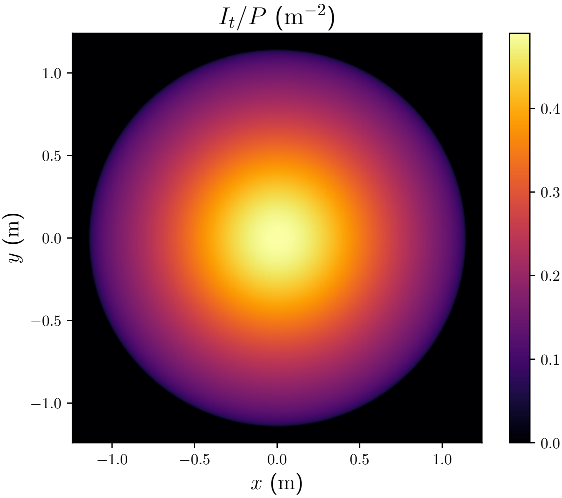


Figure 4: Plot of the transmitted intensity I_t as a function of the position on the surface.

Problem 48 … thermodynamic lottery 8 points

Two Maxwell demons, Elenka and Tomáš, work in corporation called "Ensembles Ltd.". Their task is to find certain microstates. They both have an isolated system in equilibrium composed of a box with $N = 1.00 \cdot 10^7$ monatomic ideal gas particles and volume V.

Now, each of them looks into their box and checks if the system is in the desired range of microstates. If yes, these boxes are quickly shipped to the customers. The employee with higher number of shipped boxes gets a promotion. Elenka really wants to be promoted, so she heated up Tomáš's box just a little (she is a demon after all).

Consider that the desired range of microstates is the same for both of them and that each microstate compatible with the macroscopic quantities is equally probable. Furthermore, *Elenka's box is at temperature $T_\mathrm{e} = T$ and Tomáš's is at $T_\mathrm{t} = 1.000\,01T$, what is the ratio between the probability that Elenka looks and finds a desired microstate and the probability that Tomáš does? Consider a classical setting (T and N are large enough).*

Marek J. wants to be a Maxwell's demon.

The key ability of Maxwell's demon is to perceive the world on two levels – microscopic and macroscopic. The macroscopic level is also familiar to us, human beings. A macroscopic state, in this case, is a set of physical quantities emerging from a collection of many particles, which somehow characterizes the state of our system. These include pressure, temperature, volume, density, etc. Think about how these quantities lose their meaning as we look closer and closer at the box of gas (and in very brief moments). In other words, we look from the perspective of a particle. We can describe a point particle by its position and momentum (two sets of three numbers), and within classical mechanics (e.g., in the language of Newton's laws), we have fully described its state (we have all the necessary information to describe its time evolution). A microstate is thus the set of positions and momenta of all particles (in other words, a point in a $6N$ dimensional space). How many microstates are accessible to the systems/boxes of gas in the problem?

Each particle can only exist within the volume of the box, and a similar constraint applies to the particles' momentum because the gas system has a certain constant energy value $E = \Sigma_{i=1}^{N}(p_{ix}^2+p_{iy}^2+p_{iz}^2)/2m$ (a value determined with some degree of precision). After a moment of thinking about the $6N$ dimensional space, we realize that we have just described a hypercylinder with an inner cavity. The possible microstates we need to consider thus occupy a certain finite volume in phase space. However, this volume still represents a continuous/infinite number of microstates. Physicists dealt with this by discretizing this volume – dividing it into small cubes, where the number of possible microstates represents the number of these cubes.[5]

However, we won't dwell on this too much, as the problem statement mentions the desired range of microstates μ, and whether this range is continuous or discrete does not matter much in this case. The probability of finding a microstate within the given range, under the problem's assumptions, is the ratio between this range and the total number/volume of microstates accessible to the system Ω:

$$P(\mu_\mathrm{s} \in \mu) = \frac{\mu}{\Omega}\,. \tag{17}$$

We are interested in the ratio of probabilities:

$$\frac{P(\mathrm{Elenka})}{P(\mathrm{Tomas})} = \frac{\Omega_\mathrm{t}}{\Omega_\mathrm{e}}\,, \tag{18}$$

where the interval of microstates cancels out (as does the discretization of the volume in phase space).

The required ratio of volumes can be calculated directly (by integrating in $6N$ dimensional space) or by using thermodynamics (macroscopic laws) and its translation to and from microscopic laws: statistical physics (by the way, a combination of both approaches is available, for example, here: `https://en.wikipedia.org/wiki/Gibbs_paradox#Calculating\_the\_entropy_of_ideal_gas,_and_making_it\_extensive`). From statistical physics, we need Boltzmann's entropy relation:

$$S(N,V,U) = k\ln(\Omega(N,V,U(T)))\,, \tag{19}$$

[5] When starting from a quantum mechanical description, we find in the classical limit that the volume of these cubes is Planck's constant cubed. This is a consequence of the Heisenberg uncertainty principle, as within this volume, we can no longer distinguish between different states.

where k is Boltzmann's constant, and U is the internal energy of the gas, with $U = 3NkT/2$ in our case. Another key relation is the equation for the entropy of an ideal gas[6]:

$$S = Nk\left(\ln\left(\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right) + \frac{5}{2}\right) . \tag{20}$$

For the thermodynamic derivation of (20), see, for example, the classic textbook Thermodynamics and an Introduction to Thermostatistics by H. B. Callen, chapter "3.4 The Simple Ideal Gas and Multicomponent Simple Ideal Gases" (but we know that it's still the logarithm of the volume of compatible microstates).
Combining relations (17), (18), and (19), we obtain the expression for the required ratio:

$$\frac{\Omega_\mathrm{t}}{\Omega_\mathrm{e}} = e^{(S_\mathrm{t}-S_\mathrm{e})/k} .$$

We only need to express the entropy difference, with the two systems differing only in temperature:

$$S_\mathrm{t} - S_\mathrm{e} = \frac{3}{2}Nk\ln\left(\frac{T_\mathrm{t}}{T_\mathrm{e}}\right) .$$

For the required probability ratio, we then get:

$$\frac{P(\text{Elenka})}{P(\text{Tomas})} = \left(\frac{T_\mathrm{t}}{T_\mathrm{e}}\right)^{3N/2} = (1.000\,01)^{3N/2} \doteq 1.39\cdot 10^{65} .$$

We see that even though Elenka changed the temperature of Tomáš's system by only a little, and the number of particles is relatively small (compared to 10^{23}), this change led to an enormous increase in the number of new microstates in Tomáš's system, making it almost certain that Elenka will get the promotion.

Problem 49 ... giant metronome 9 points

Consider a metronome consisting of two rods. The longer rod is homogeneous, with a length of $L = 25.0\,\mathrm{m}$ and a linear density of $\lambda = 100\,\mathrm{kg{\cdot}m^{-1}}$, and is fixed at one end at a point around which it can freely rotate. The shorter rod, with a length of $l = 10.0\,\mathrm{m}$, is massless and its upper end is attached to the midpoint of the longer rod, such that it can freely rotate around the point of connection. Moreover, the shorter rod always passes through a point located at a height of $h = 11.0\,\mathrm{m}$ above the point where the lower end of the longer rod is fixed – at this point, it can both rotate and slide. At the other end of the shorter rod, there is a weight with a mass of $M = 2.00\,\mathrm{t}$. What is the period of small oscillations?

Lego was at the Prague metronome.

Solving this problem using forces would be unnecessarily difficult due to the position of the point where the shorter rod is supported. Finding out the force by which it acts and where might be interesting but complicated. And we are not even speaking about the points around which the rods are rotating. The only option left is solving the problem through the system's energy.

[6] `https://en.wikipedia.org/wiki/Sackur-Tetrode_equation`

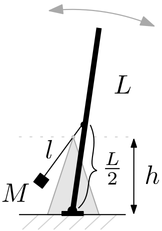


Figure 5: Diagram of the metronome.

Let us designate the angle by which the longer rod deviates from the vertical direction as φ. The rod's weight is $m_1 = L\lambda$, located in its point of mass, i.e. on the circular arc in the distance $L/2$ from the point around which the rod is rotating. Its potential energy will then be $E_{\mathrm{p1}} = (g/2)L^2\lambda\cos\varphi$.

We calculate its angular momentum as the angular momentum of a homogeneous rod with respect to its end

$$I = \frac{1}{3}m_1L^2 = \frac{1}{3}\lambda L^3 \,.$$

Then, its kinetic energy is

$$E_{\mathrm{k1}} = \frac{1}{2}I\ \omega_1^2 = \frac{1}{6}\lambda L^3\dot{\varphi}^2 \,.$$

The movement of the shorter rod is more complex. To describe it in general would once again be difficult. However, we can focus on the small oscillations. For small deviations $\varphi \ll 1$, we can expand the sine and cosine as $\sin\varphi \approx \varphi, \cos\varphi \approx 1-\varphi^2/2$. The coordinates of the center of the long rod are then

$$x_1 \approx \frac{L}{2}\varphi, y_1 \approx \frac{L}{2}\left(1-\frac{\varphi^2}{2}\right) \,.$$

Let us calculate the distance between the center of the longer rod and the point where the shorter rod is supported. Unsurprisingly, we will use the Pythagorean theorem, except we will neglect some factors. The coordinates of the point supporting the shorter rod are $[0, h]$

$$\begin{aligned} l_1^2 = (y_1-h)^2 + x_1^2 &= \left(\frac{L}{2} - h - \frac{L}{2}\frac{\varphi^2}{2}\right)^2 + \left(\frac{L}{2}\varphi\right)^2 \\ &= \left(\frac{L}{2}-h\right)^2 - \left(\frac{L}{2}-h\right)\frac{L}{2}\varphi^2 + \frac{L^2\varphi^2}{4} = \left(\frac{L}{2}-h\right)^2 + h\frac{L}{2}\varphi^2 \,, \end{aligned}$$

meaning that

$$l_1 = \sqrt{\left(\frac{L}{2}-h\right)^2 + h\frac{L}{2}\varphi^2} = \left(\frac{L}{2}-h\right)\sqrt{1+\frac{h\frac{L}{2}}{\left(\frac{L}{2}-h\right)^2}\varphi^2} = \left(\frac{L}{2}-h\right) + \frac{hL}{4\left(\frac{L}{2}-h\right)}\varphi^2 \,.$$

Let us also designate

$$l_0 = l_1|_{\varphi=0} = \frac{L}{2} - h \,.$$

Now, we can calculate the height of the weight which is at the end of the rod with respect to φ. In particular, we designate Δy as the height difference by which the weight is lower with respect to the point at which the rod is supported. Then its y coordinate is $y_2 = h - \Delta y$. Since the rod is straight, we can use the triangle similarity to put into an equation the ratio of the length of the rod above this point to the distance of the upper end above this point $(y_1 - h)$ and the ratio of the length below this point to Δy

$$\frac{l_1}{l_0 - \frac{L}{2}\frac{\varphi^2}{2}} = \frac{l - l_1}{\Delta y}$$

$$\Delta y = \left(l_0 - \frac{L}{2}\frac{\varphi^2}{2}\right)\frac{l - l_1}{l_1}\,.$$

Let us examine further the last fraction

$$\frac{l - l_0 - \frac{hL}{4l_0}\varphi^2}{l_0 + \frac{hL}{4l_0}\varphi^2} = \frac{1}{l_0}\frac{l - l_0 - \frac{hL}{4l_0}\varphi^2}{1 + \frac{hL}{4l_0^2}\varphi^2} \approx$$

$$\approx \frac{l - l_0}{l_0} - \frac{hL}{4l_0^2}\varphi^2\left(1 + \frac{l - l_0}{l_0}\right) = \frac{l - l_0}{l_0} - \frac{hLl}{4l_0^3}\varphi^2\,.$$

By substituting we get

$$\Delta y = \left(l_0 - \frac{L}{2}\frac{\varphi^2}{2}\right)\frac{l - l_0}{l_0} - \frac{hLl}{4l_0^3}\varphi^2$$

$$\approx l - l_0 - \frac{hLl}{4l_0^2}\varphi^2 - L\frac{l - l_0}{4l_0}\varphi^2$$

and substitute the result into our expression for the height of the weight $y_2 = h - \Delta y$.

$$y_2 = \frac{L}{2} - l + \frac{hLl}{4l_0^2}\varphi^2 + L\frac{l - l_0}{4l_0}\varphi^2$$

We can see that for $\varphi = 0$, we get a height $L/2 - l$, which agrees. From the rest, we can calculate the contribution to the change in potential energy from the weight located at the end

$$E_{\mathrm{p2}} = Mg\Delta y_2(\varphi) = \frac{1}{2}Mg\left(\frac{hLl}{2l_0^2} + L\frac{l - l_0}{2l_0}\right)\varphi^2\,.$$

Regarding the kinetic energy of the weight, this direction is negligible as the movement in the x axis direction is more significant. From the triangle similarity, we express x_2

$$\frac{l_1}{x_1} = \frac{l - l_1}{-x_2}$$

$$-x_2 = \frac{L}{2}\varphi\frac{l - l_1}{l_1}\,.$$

For the fraction, we can substitute from the previous part. We even know that we do not have to care about the term with φ^3, so substituting the order zero is enough

$$-x_2 = \frac{L}{2}\frac{l - l_0}{l_0}\varphi\,,$$

where the minus sign only represents the deviation in the opposite direction. That does not have to concern us as we only care about the square of the velocity with respect to the time derivative $\dot{\varphi}$

$$v_2^2 = \frac{L^2}{4}\frac{(l-l_0)^2}{l_0^2}\dot{\varphi}^2 \,.$$

From here, we obtain the kinetic energy of the weight as

$$E_{\mathrm{k2}} = \frac{1}{2}Mv_2^2 = \frac{1}{2}M\frac{L^2}{4}\frac{(l-l_0)^2}{l_0^2}\dot{\varphi}^2 \,.$$

Now, we only have to compute E_{p1} into an analogical form. We only care about the change of the potential energy with respect to the state $\varphi = 0$, particularly the second term of its Taylor series. By expanding the cosine, we get

$$\Delta E_{\mathrm{p1}} = E_{\mathrm{p1}}(\varphi) - E_{\mathrm{p1}}(0) = \frac{1}{2}gL^2\lambda\left(1-\frac{\varphi^2}{2}\right) - \frac{1}{2}gL^2\lambda = -\frac{1}{4}gL^2\lambda\varphi^2 \,.$$

The minus sign is again important and understandable in the expression: $\varphi = 0$ would be an unstable position for the rod alone.

The total potential energy can be calculated as

$$E_{\mathrm{p}} = \Delta E_{\mathrm{p1}} + E_{\mathrm{p2}} = \frac{1}{2}g\left(M\frac{hLl}{2l_0^2} + ML\frac{l-l_0}{2l_0} - \frac{1}{2}L^2\lambda\right)\varphi^2 \,,$$

therefore we obtain the “effective rigidity”

$$k_{\mathrm{ef}} = g\left(M\frac{hLl}{2l_0^2} + ML\frac{l-l_0}{2l_0} - \frac{1}{2}L^2\lambda\right)$$

as the rigidity for the variable φ but in different units then for a classical rigidity.

Similarly, from the kinetic energy, we find the effective mass

$$E_{\mathrm{k}} = E_{\mathrm{k1}} + E_{\mathrm{k2}} = \frac{1}{2}\left(M\frac{L^2}{4}\frac{(l-l_0)^2}{l_0^2} + \frac{1}{3}\lambda L^3\right)\dot{\varphi}^2$$

$$m_{\mathrm{ef}} = M\frac{L^2}{4}\frac{(l-l_0)^2}{l_0^2} + \frac{1}{3}\lambda L^3 \,.$$

This value also is not be in kilograms, but is only important that the “effective rigidity and mass” are related to the particular variables, φ, and $\dot{\varphi}$ in our case.

Finally, we substitute the results into the equation for a period of an oscillator

$$T = 2\pi\sqrt{\frac{m_{\mathrm{ef}}}{k_{\mathrm{ef}}}} = 2\pi\sqrt{\frac{1}{g}\frac{M\frac{L^2}{4}\frac{(l-l_0)^2}{l_0^2} + \frac{1}{3}\lambda L^3}{M\frac{hLl}{2l_0^2} + ML\frac{l-l_0}{2l_0} - \frac{1}{2}L^2\lambda}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{\frac{M}{4}\frac{(l-l_0)^2}{l_0^2} + \frac{1}{3}\lambda L}{M\frac{hl}{2l_0^2} + M\frac{l-l_0}{2l_0} - \frac{1}{2}L\lambda}} = 5.65\,\mathrm{s}\,.$$

Problem 50 ... charged hoop in a sphere

9 points

Consider a hollow, ideally conducting sphere of radius $R = 9.0\,\mathrm{m}$. A point charge $Q = 6.0\,\mathrm{nC}$ is placed at its center. Next, a circular ring of radius $\rho = 7.0\,\mathrm{m}$ is placed inside the sphere so that its axis passes through the center of the sphere and the distance between the center of the sphere and the center of the ring is $r = 5.0\,\mathrm{m}$. The ring is uniformly charged with linear charge density $\lambda = 3.0\,\mathrm{\mu C\cdot m^{-1}}$. A test charge with charge-to-mass ratio $q/m = 0.30\,\mathrm{C\cdot kg^{-1}}$ (assumed not to affect the charge distribution on the sphere) is placed on the line segment connecting the centers of the sphere and the ring. The test charge is allowed to stabilize at its equilibrium position inside the sphere. If the test charge is then slightly displaced, what will be the period T of its small oscillations? *Petr tried to upcycle his Fyziklani problem from last year.*

To begin solving the problem, we must first understand what it means that the sphere is (ideally) conducting – if we place an external charge inside the sphere, all charges on its surface rearrange so that the total potential on the sphere is constant. This, however, means that a surface charge density appears on the sphere, which produces its own electric field. A classic trick of electrostatics is that for simple objects such as a plane or a sphere, we can simulate the effect of this field by inserting an image charge of a certain size at a certain location. For a sphere, we can find the image charge using spherical inversion. If we have a charge q inside a conducting sphere of radius R at a distance l from the center, the correct image charge is

$$q' = -\frac{R}{l}q\,,$$

located at a distance

$$l' = \frac{R^2}{l}$$

from the center of the sphere along the axis connecting the sphere's center and the charge q. The electric field inside the sphere can then be determined simply by summing the contributions from all real and image charges.[7]

The charge at the center of the sphere is symmetrically placed with respect to its surface, so it automatically induces a constant surface charge density (and since the sphere is not grounded, this constant must be zero) – we are therefore lucky and do not need the image-charge trick for this case.

For the ring, however, image charges are unavoidable – using spherical inversion, we find the image ring. Imagine the ring is composed of infinitesimal charges $\mathrm{d}q = \lambda\,\mathrm{d}l$ located at a distance $l = \sqrt{r^2+\rho^2}$ from the sphere's center. These can be inverted, but we must also take into account that the ring stretches, which reduces the charge density:

$$\lambda' = \frac{\mathrm{d}q'}{\mathrm{d}l'} = \frac{-\frac{R}{\sqrt{r^2+\rho^2}}\,\mathrm{d}q}{\frac{P}{\rho}\,\mathrm{d}l} = -\frac{R\rho}{P\sqrt{r^2+\rho^2}}\lambda\,,$$

where P denotes the radius of the image ring. The distance of its points from the center is

$$L = \frac{R^2}{\sqrt{r^2+\rho^2}}\,.$$

[7] These two relations can be derived solely from the requirement of a constant potential on the sphere's surface.

Now, let us write the electric field on the axis of the ring passing through the center of the sphere. For the charge Q at the center, we obtain the field simply from Coulomb's law:

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}\,,$$

where z is the distance from the center along the axis. The field generated by the ring will be slightly more complicated. Coulomb's law can be used for each infinitesimal charge $\mathrm{d}q$,

$$\mathrm{d}E_2 = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{\rho^2 + (r-z)^2}\,\mathrm{d}l\,.$$

The total field is then obtained by integrating around the entire ring. However, the integration can be simplified using two facts. First, by symmetry along the axis, all components of the ring's field except the axial one cancel out. Thus, we only need the z–component:

$$\mathrm{d}E_{2,z} = \mathrm{d}E_2 \cos\vartheta\,,$$

where

$$\cos\vartheta = \frac{r-z}{\sqrt{\rho^2 + (r-z)^2}}\,.$$

Second, the contribution of each $\mathrm{d}q$ is the same up to direction, so the integral reduces to multiplication by the factor $2\pi\rho$. Altogether,

$$E_2 = \frac{\lambda}{2\varepsilon_0}\frac{\rho(r-z)}{\left(\rho^2 + (r-z)^2\right)^{3/2}}\,.$$

Similarly, we can determine the contribution from the image ring. Before writing the resulting field, let us denote the distance of the image ring from z by d and its radius by P:

$$d = \frac{R^2 r}{r^2+\rho^2} - z\,,$$
$$\mathrm{P} = \frac{R^2\rho}{r^2+\rho^2}\,.$$

These relations were derived using triangle similarity between the real ring and the axis, and the image ring and the axis. The total contribution from the image ring is then

$$\begin{aligned} E_3 &= -\frac{\lambda}{2\varepsilon_0}\frac{R\rho}{\left(r^2+\rho^2\right)^{3/2}}\frac{R^2 r - z\left(r^2+\rho^2\right)}{\left(\left(\frac{R^2 r}{r^2+\rho^2} - z\right)^2 + \frac{R^4\rho^2}{\left(r^2+\rho^2\right)^2}\right)^{3/2}} \\ &= -\frac{\lambda R\rho\left(r^2+\rho^2\right)^{3/2}}{2\varepsilon_0}\frac{R^2 r - z\left(r^2+\rho^2\right)}{\left(\left(R^2 r - z\left(r^2+\rho^2\right)\right)^2 + R^4\rho^2\right)^{3/2}}\,. \end{aligned}$$

The total field along the axis is therefore

$$E = E_1 + E_2 + E_3\,,$$

and the equilibrium position z_0 is obtained by solving $E(z_0) = 0$. The resulting algebraic nightmare is best solved numerically, e.g., in Python – for instance, we used the function bisect from the optimize module of the scipy library. We obtain the root

$$z_0 \doteq 3.383\,\mathrm{m}.$$

Thus, the equilibrium position of the test charge is located a little over three meters above the center of the sphere on the axis of the ring.

How do we now find the oscillation period? First, we write the equation of motion of the test charge at position z:

$$m\ddot{\xi} = qE(\xi)\,.$$

For simplicity of notation, we have introduced a new variable $\xi = (z - z_0)$ (note that any derivative of ξ is equal to the corresponding derivative of z, since z_0 is a constant). If the field were linear, i.e. $E(\xi) = -\varepsilon\xi$, where ε is some positive constant with units $\mathrm{V{\cdot}m^{-2}}$, we would be done, because we could transform the equation of motion into the form

$$\ddot{\xi} + \frac{q}{m}\varepsilon\xi = 0\,,$$

which is identical to the equation of harmonic oscillations

$$\ddot{x} + \omega^2 x = 0\,.$$

The period T can then be obtained from the angular frequency ω using the well-known relation $T = 2\pi/\omega$. The electric field E itself is not linear, but for small displacements we can expand it in a Taylor series and keep the first two terms. The simplest approach is to perform a numerical derivative, but we can compute it analytically as well. The expansion has the form

$$E(z) = E\left(z_0\right) + \left.\frac{\partial E}{\partial z}\right|_{z_0}\left(z - z_0\right) + o\left(\left(z - z_0\right)^2\right) = \left.\frac{\partial E}{\partial z}\right|_{z_0}\left(z - z_0\right) + o\left(\left(z - z_0\right)^2\right)\,,$$

where the first term is zero because we expand around the equilibrium position. The derivative of the electric field with respect to z is computed as the sum of the derivatives of the individual field contributions:

$$\begin{aligned}
\frac{\partial E_1}{\partial z} &= -\frac{1}{2\pi\varepsilon_0}\frac{Q}{z^3}\,,\\
\frac{\partial E_2}{\partial z} &= \frac{\lambda\rho}{2\varepsilon_0}\left(\frac{2\left(r - z\right)^2 - \rho^2}{\left(\rho^2 + \left(r - z\right)^2\right)^{5/2}}\right)\,,\\
\frac{\partial E_3}{\partial z} &= -\frac{\lambda}{2\varepsilon_0}\frac{R\rho}{\left(r^2 + \rho^2\right)^{5/2}}\left(\frac{2\left(R^2 r - z\left(r^2 + \rho^2\right)\right)^2 - R^4\rho^2}{\left(\left(\frac{R^2 r}{r^2+\rho^2} - z\right)^2 + \frac{R^4\rho^2}{\left(r^2+\rho^2\right)^2}\right)^{5/2}}\right)\,,\\
\frac{\partial E}{\partial z} &= \frac{\partial E_1}{\partial z} + \frac{\partial E_2}{\partial z} + \frac{\partial E_3}{\partial z}\,.
\end{aligned}$$

To evaluate the derivative at the point z_0, we may again use Python. We obtain the intermediate result

$$\left.\frac{\partial E}{\partial z}\right|_{z_0} \doteq -753\,\mathrm{V{\cdot}m^{-2}} \quad \text{respektive} \quad \varepsilon \doteq 753\,\mathrm{V{\cdot}m^{-2}}\,.$$

The period T is then computed as

$$T = \frac{2\pi}{\sqrt{\frac{q}{m}\varepsilon}} \doteq 0.42\,\mathrm{s}\,.$$

Problem 51 ... Easterly

8 points

Kuba was successful on Easter Monday; because of that, a heavy cluster of ribbons was located at the end of his Easter whip. Determine the weight of this cluster, assuming the willow whip is a perfectly thin, massless, homogeneous rod of length $l = 1.0\,\mathrm{m}$ (with ends A and B) and bending stiffness $K = 0.80\,\mathrm{N{\cdot}m^2}$; when the whip was held vertically at the end A (with the ribbons tied to the end B), it was deflected from the vertical axis by an angle $\theta_0 = 45\,°$, and the tangent at the free end B of the whip made an angle θ_0 with the vertical, while the tangent at the held end A was vertical. The bending stiffness is defined by $\tau = K\ \kappa$, where τ is the external bending moment and κ is the rod curvature.

Kuba was inspired by the traditional mistreatment of women.

Let the Easter whip be placed at rest such that its fixed end lies at the origin of the coordinate system $A = (0, 0)$, and its free end at the point $B = (0, l)$. We may divide the Easter whip into infinitesimal length elements of constant length $\mathrm{d}s$, bounded by points A_k and A_{k+1}, which behave as massless rigid rods. The index k increases from the fixed end to the free one, so $A_0 = A$.

Upon deformation, the points A_k map to points A'_k, thereby creating a nonzero angle between adjacent elements. We denote $\mathrm{d}\mathbf{r}_k = A'_{k+1} - A'_k$. The element E_k (i.e., the element with vector $\mathrm{d}\mathbf{r}_k$) evidently exerts a reaction force $\mathbf{R}_{k+1}$ on the element E_{k+1}. Besides these reaction forces, only a pure torque arising from the elastic response to the nonzero angle acts on the elements. Thus, on element E_k acts the force $\mathbf{R}_k$ at point A_k from element E_{k-1}, and simultaneously the force $-\mathbf{R}_{k+1}$ from element E_{k+1}. It is true that for an element E_k (due to the conservation of momentum principle) that

$$\mathbf{R}_k - \mathbf{R}_{k+1} = 0\,. \tag{21}$$

We also know that no reaction force acts on the free end; instead, an external force $\mathbf{F}$ acts there. Hence,

$$\mathbf{R}_n + \mathbf{F} = 0\,, \tag{22}$$

where n is the index of the last element. Combining (21) and (22), we obtain $\mathbf{R}_k = -\mathbf{F}$ for all k.

Next, we consider the torques acting on element E_k with respect to the axis of rotation at point A_k. On the element acts the torque $\boldsymbol{\tau}_k$ due to elastic bending at A_k, and also the torque $-\boldsymbol{\tau}_{k+1}$ due to elastic bending at A_{k+1}. Since both torques are pure (i.e., they produce no net force on the element), the vectors $\boldsymbol{\tau}_k$ are identical with respect to any axis of rotation.

Furthermore, the element is subject to a torque generated by the reaction forces $\mathbf{R}_k$ and $-\mathbf{R}_{k+1}$. The first of these has zero lever arm with respect to the axis at A_k and therefore contributes nothing. Euler's second law can be applied as

$$\boldsymbol{\tau}_k - \boldsymbol{\tau}_{k+1} - \mathrm{d}\mathbf{r}_k \times \mathbf{R}_{k+1} = 0\,.$$

The infinitesimal vectors $\mathrm{d}\mathbf{r}_k$ must all have equal magnitude $\mathrm{d}s$, corresponding to the length of one element in the undeformed Easter whip (the elements do not deform, but only rotate, since we neglect stretching). We may therefore write $\mathrm{d}\mathbf{r}_k = \mathbf{t}_k\,\mathrm{d}s$, where $\mathbf{t}_k$ is the unit vector along $\mathrm{d}\mathbf{r}_k$. Substituting $\mathbf{R}_{k+1} = -\mathbf{F}$, we obtain

$$-\,\mathrm{d}\boldsymbol{\tau}_k + \mathrm{d}s\,\mathbf{t}_k \times \mathbf{F} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s} = \mathbf{t} \times \mathbf{F}\,, \tag{23}$$

where $\mathbf{t}_k$ becomes the tangent vector $\mathbf{t}$. Let $\varphi(s)$ be the angle between $\mathbf{t}(s)$ and $\mathbf{e}_x$, in the sense that

$$\mathbf{t} = \cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y\,.$$

The curvature of the Easter whip at point s is defined as

$$\begin{aligned}\kappa &= \left\|\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\right\| = \left\|\frac{\mathrm{d}}{\mathrm{d}s}\left(\cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y\right)\right\| = \left\|-\sin\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}s}\mathbf{e}_x + \cos\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}s}\mathbf{e}_y\right\| \\ &= \left[\sin^2\varphi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 + \cos^2\varphi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\right]^{1/2} = \left|\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right|\,.\end{aligned}$$

We are also given the bending stiffness K via

$$\tau = K\kappa \quad \Rightarrow \quad \boldsymbol{\tau} = -K\frac{\mathrm{d}\varphi}{\mathrm{d}s}\mathbf{e}_z\,, \tag{24}$$

which follows directly from the direction of the torque produced by a change $\mathrm{d}\varphi$. The cross product on the right-hand side of (23) may be rewritten as

$$\mathbf{t} \times \mathbf{F} = \left(\cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y\right) \times \left(-mg\,\mathbf{e}_y\right) = -mg\cos\varphi\,\mathbf{e}_z\,.$$

Substituting (24) into (23) yields

$$-K\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} = -mg\cos\varphi \quad \Rightarrow \quad \frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} = \frac{mg}{K}\cos\varphi\,.$$

Now let $\theta = (\pi/2) - \varphi$, which is more natural for our problem, since θ corresponds to the rightward deflection from the vertical axis. Then

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -\frac{mg}{K}\sin\theta\,. \tag{25}$$

But this equation is familiar! It has the same form as the equation of motion of a pendulum

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L}\sin\theta = -\alpha\sin\theta\,,$$

where θ is the angular displacement from the vertical, L is the length of the pendulum, and the ratio is denoted α.

We may now discuss the solution using the velocity phase space of the pendulum. For fixed ribbon mass m and stiffness K, the parameter α is constant. The boundary conditions imply that the trajectory in the phase space must begin on the axis $\theta = 0$ and end on the axis $\dot{\theta} = 0$. This in particular means that we must not leave the region of closed trajectories, resembling an *eye* centered at $\theta = \dot{\theta} = 0$.

The length of the Easter whip therefore corresponds to one quarter of the period of the notional pendulum, which can be expressed as

$$T = \frac{4}{\sqrt{\alpha}}\mathsf{K}\left(\sin\frac{\theta_0}{2}\right) \quad \Rightarrow \quad l = \frac{1}{\sqrt{\alpha}}\mathsf{K}\left(\sin\frac{\theta_0}{2}\right), \tag{26}$$

where $\theta_0 = \theta(l)$ (corresponding to the maximal deflection of the pendulum) and $\mathsf{K}(k)$ is the complete elliptic integral of the first kind, which can be derived by solving (25) or simply looked up.

The function $\mathsf{K}(k) \to \pi/2$ as $k \to 0$. For small θ we therefore obtain the period

$$T \approx 2\pi\frac{1}{\sqrt{\alpha}} = 2\pi\sqrt{\frac{L}{g}},$$

which matches the linear approximation. Conversely, $\mathsf{K}(k) \to \infty$ as $k \to 1$, which occurs for $\theta_0 = \pi$. It thus appears that the admissible values of the Easter whip length l lie in the interval

$$l \in \left[\frac{\pi}{2\sqrt{\alpha}}, \infty\right) = \left[\frac{\pi}{2}\sqrt{\frac{K}{mg}}, \infty\right) =: [l_c, \infty)\,.$$

But what happens for $l < l_c$? Note that the function $\theta(s) \equiv 0$ is always an admissible solution satisfying the boundary conditions. For $l > l_c$ such a solution is unstable, and moreover we are interested in a solution in which the Easter whip is actually bent. For $l < l_c$, however, this trivial solution is the *only* solution. For a very short Easter whip, or equivalently a very small force, the Easter whip thus remains undeformed. For $l > l_c$ we can always find a corresponding deformed solution. The problem asks for the mass of the ribbons. Using (26), we calculate l as

$$l = \sqrt{\frac{K}{mg}}\mathsf{K}\left(\sin\frac{\theta_0}{2}\right) \quad \Rightarrow \quad m = \frac{K}{gl^2}\mathsf{K}^2\left(\sin\frac{\theta_0}{2}\right) \doteq 0.22\,\mathrm{kg}\,.$$

Problem 52 ... thermal neutrons — 9 points

Jindra wants to detect thermal neutrons more than anything else in the world. For this reason, he acquired a semiconductor pixel particle detector and lithium fluoride LiF. Jindra's detector is a semiconductor slab with a layer of lithium fluoride on its top surface. Thermal neutrons hit the layer of lithium fluoride perpendicularly.

The lithium isotope ^{6}Li reacts with thermal neutrons to produce an alpha particle and a tritium nucleus

$$^6\mathrm{Li} + \mathrm{n} \longrightarrow {}^4\mathrm{He} + {}^3\mathrm{H}.$$

The cross-section of the reaction is $\sigma = 940\,\mathrm{b}$. The tritium nucleus is emitted with a kinetic energy of $2.73\,\mathrm{MeV}$*, and the alpha particle is emitted with a kinetic energy of* $2.05\,\mathrm{MeV}$*. Jindra's* *detector indirectly detects neutrons by detecting alpha particles and tritium nuclei. The number of direct interactions of neutrons in the semiconductor is negligible. You may assume that every alpha particle or tritium nucleus entering the semiconductor detector is detected. The range of an alpha particle with energy* $2.05\,\mathrm{MeV}$ *in lithium fluoride is* $l_\alpha = 5.94\,\mathrm{\mu m}$. *The range of a tritium nucleus with energy* $2.73\,\mathrm{MeV}$ *in lithium fluoride is* $l_\mathrm{H} = 32.9\,\mathrm{\mu m}$. *The particle paths are straight and do not bend.*

Jindra wants to find such a thickness of the LiF layer that he achieves the highest possible neutron detection efficiency (i.e., the number of detected neutrons divided by the number of incident neutrons), yet at the same time, he does not want to use more LiF than necessary. What is the highest achievable detection efficiency?

The density of lithium fluoride is $\rho = 2\,635\,\mathrm{kg{\cdot}m^{-3}}$. *Lithium contains* $p = 7.50\,\%$ *of the isotope* ^{6}*Li atoms. The rest are atoms of the isotope* ^{7}*Li, which do not react with neutrons.*

Jindra has not yet acquired a thermal-neutron detector, ho only modelelled the problem in Geant4.

First, let us discuss some facts about the dynamics of the reaction of the isotope ^{6}Li with a thermal neutron

$$^6\mathrm{Li} + \mathrm{n} \longrightarrow {}^4\mathrm{He} + {}^3\mathrm{H}\,.$$

The lithium atom is trapped in the crystal lattice of lithium fluoride, and due to the equipartition theorem, it has a kinetic energy on the order of $10^{-2}\,\mathrm{eV}$, just like the thermal neutron. The kinetic energies of the alpha particle and the tritium nucleus are several orders of magnitude higher. Their kinetic energy originates from the converted binding energy of the lithium nucleus. In a laboratory frame of reference, the center of mass of the ^{6}Li and neutron system moves orders of magnitude slower than the speed of the emerging alpha particle and tritium nucleus, so it can be considered stationary. Since this is a two-particle decay of a system with zero initial velocity, the products ^{4}He and ^{3}H always have the same kinetic energy in the laboratory reference frame. The decay does not favor any direction; alpha particles are emitted isotropically, and the tritium nucleus always moves in the opposite direction to the alpha particle. The alpha particles and the tritium nuclei are charged, so they lose energy mainly through electromagnetic interaction with electrons in lithium fluoride. At the same time, they are several orders of magnitude heavier than electrons, so their direction changes very little during interactions, and they travel straight. Due to the same initial energy during each reaction of a neutron with a ^{6}Li nucleus, both the alpha particles and the tritium nuclei have constant range distances in the lithium fluoride, $l_\alpha = 5.94\,\mathrm{\mu m}$ and $l_\mathrm{H} = 32.9\,\mathrm{\mu m}$.

The probability density that an alpha particle flies into the solid angle $\mathrm{d}\Omega$ is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{4\pi}\,.$$

If we express the infinitesimal solid angle $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ in terms of the azimuthal angle φ and the zenith angle θ, we get the probability density for the angles φ and θ

$$f(\theta,\varphi) = \frac{\mathrm{d}^2 P}{\mathrm{d}\theta\,\mathrm{d}\varphi} = \frac{1}{4\pi}\sin\theta\,.$$

To have any chance of detecting the alpha particle, the neutron must interact with the lithium isotope at most l_α above the surface of the semiconductor detector. Similarly, to detect the tritium nucleus, the neutron must interact at most l_H above the surface of the semiconductor. Since tritium and alpha particles always fly in opposite directions, there is no situation in which we would detect both particles simultaneously. The probability of detecting both decay products $P\left(\alpha \wedge {}^3\mathrm{H}\right) = 0$ is zero. Either we detect the alpha particle, or we detect the tritium nucleus, or we detect nothing. Thus, the probability $P\left(\alpha \vee {}^3\mathrm{H}\right)$ of detecting an alpha particle or tritium is simply the sum

$$P\left(\alpha \vee {}^3\mathrm{H}\right) = P(\alpha) + P\left({}^3\mathrm{H}\right) - P\left(\alpha \wedge {}^3\mathrm{H}\right) = P(\alpha) + P\left({}^3\mathrm{H}\right).$$

Given the symmetry of the situation, we only need to derive the probability of detecting the alpha particle $P(\alpha)$. The probability of detecting tritium $P\left({}^3\mathrm{H}\right)$ is obtained by simply substituting $l_\alpha \to l_\mathrm{H}$.

Let a thermal neutron react with the lithium nucleus at a height x above the surface of the semiconductor. The range of the alpha particle in lithium fluoride is limited to a spherical surface with a radius l_α. For $x > l_\alpha$, the probability of detecting the alpha particle in the detector is zero – $P(\alpha \mid x) = 0$. For $x \leq l_\alpha$, we obtain the conditional probability of detecting the alpha particle by integrating over the solid angle on the spherical surface, where the alpha particle passes through the semiconductor surface. Orient the Cartesian coordinate axis z from the interaction point towards the detector, perpendicular to the semiconductor surface. The zenith angle θ is measured from this axis. The alpha particle will hit the detector if it leaves the interaction point with a zenith angle between 0 and $\theta_\mathrm{max} = \arccos(x/l_\alpha)$. The integral for the conditional probability is

$$P(\alpha \mid x) = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\theta_\mathrm{max}} f(\theta,\varphi)\,\mathrm{d}\theta = 2\pi \int_0^{\theta_\mathrm{max}} \frac{1}{4\pi} \sin\theta\,\mathrm{d}\theta = \frac{1}{2}\left[-\cos\theta\right]_0^{\arccos(x/l_\alpha)},$$
$$P(\alpha \mid x) = \frac{1}{2}\left(1 - \frac{x}{l_\alpha}\right).$$

The general relationship for the conditional probability of detection as a function of x is

$$P(\alpha \mid x) = \begin{cases} \frac{1}{2}\left(1 - \frac{x}{l_\alpha}\right), & 0 \leq x \leq l_\alpha, \\ 0, & x > l_\alpha. \end{cases}$$

The probability that a neutron interacts above x from the semiconductor surface is

$$I(x) = 1 - \exp(-(D - x)/D_0) \approx (D - x)/D_0, \quad 0 \leq x \leq D,$$

where D is the total thickness of the lithium fluoride layer, and D_0 is the mean free path of neutrons. We assumed $D_0 \gg D$ and used the approximation $-\exp(-y) \approx y$ for $|y| \ll 1$. The probability density that a neutron interacts at a height x is

$$g(x) = \left|\frac{\mathrm{d}I}{\mathrm{d}x}\right| = \frac{1}{D_0}.$$

The total probability of detecting the alpha particle is determined using the integral

$$P(\alpha) = \int_0^\infty P(\alpha \mid x)\, g(x)\,\mathrm{d}x = \int_0^{l_\alpha} \frac{1}{2}\left(1 - \frac{x}{l_\alpha}\right)\frac{1}{D_0}\,\mathrm{d}x = \frac{1}{2D_0}\left[x - \frac{x^2}{2l_\alpha}\right]_0^{l_\alpha} = \frac{l_\alpha}{4D_0}.$$

The layer of lithium fluoride must be at least as thick as l_α. The probability of detecting the tritium nucleus is then analogous

$$P\left({}^3\mathrm{H}\right) = \frac{l_\mathrm{H}}{4D_0}\,,$$

and the layer of lithium fluoride must have a minimum thickness of l_H.

The probability of detecting a thermal neutron via its reaction with the lithium isotope ^{6}Li is

$$P\left(\alpha \vee {}^3\mathrm{H}\right) = P(\alpha) + P\left({}^3\mathrm{H}\right) = \frac{l_\alpha + l_\mathrm{H}}{4D_0}\,,$$

provided the lithium fluoride layer is thicker than the maximum of the lengths l_α and l_H. To achieve the highest detection efficiency while using only the necessary amount of LiF, the LiF layer will have a thickness of $D = l_\mathrm{H} = 32.9\,\mathrm{\mu m}$.

Now, the final task remains. To evaluate the probability $P\left(\alpha \vee {}^3\mathrm{H}\right)$, we must calculate the mean free path of neutrons D_0 in lithium fluoride. The molar mass of fluorine is $M_\mathrm{F} = 19.0\,\mathrm{g{\cdot}mol^{-1}}$. The molar masses of the isotopes ^{6}Li and ^{7}Li are $M_6 = 6.00\,\mathrm{g{\cdot}mol^{-1}}$ and $M_7 = 7.00\,\mathrm{g{\cdot}mol^{-1}}$, respectively. The molar mass of lithium fluoride LiF is

$$M_\mathrm{LiF} = M_\mathrm{F} + (pM_6 + (1-p)M_7) = 25.9\,\mathrm{g{\cdot}mol^{-1}}.$$

In every molecule of LiF, there is one lithium atom. The volumetric density of the isotope ^{6}Li in lithium fluoride is

$$n_6 = p\frac{\rho N_\mathrm{A}}{M_\mathrm{LiF}} = 4.59 \cdot 10^{27}\,\mathrm{m^{-3}},$$

where N_A is Avogadro's constant. The mean free path of neutrons in lithium fluoride is

$$D_0 = \frac{1}{\sigma n_6} = 2.32\,\mathrm{mm} \gg D\,,$$

so our initial assumption $D_0 \gg D$ was correct. The probability of detecting thermal neutrons in Jindra's detector is

$$P\left(\alpha \vee {}^3\mathrm{H}\right) = \frac{l_\alpha + l_\mathrm{H}}{4D_0} = 0.419\,\%\,.$$

Problem 53 ... attractive Petr — 8 points

As we well know, David repels women with a force that acts radially outward from him and depends only on the distance with a constant of proportionality of $120\,\mathrm{N{\cdot}m^2}$. *On the other hand, there is always a cluster of women around Petr. David wanted to measure how Petr attracts women and found that Petr's charm is a radial attractive force with a constant of proportionality of* $30.0\,\mathrm{N{\cdot}m^3}$.

Now, David is standing in a plane surface at a distance of $1.00\,\mathrm{m}$ *from Peter, when a woman appears in the same plane near them. Determine the area of the region in which she can appear so that she would be attracted to Peter, i.e., so that the resulting force acting on her would form an angle smaller than a right angle with the line connecting her and Petr. Do not consider any other forces than those mentioned.*

David was thinking about how to follow up on a problem from the Physics Brawl.

Before calculating, it is helpful to think about the problem statement qualitatively. We notice that Peter's attractive force decreases faster with increasing distance than David's repulsive force. From Peter's point of view, there is a threshold in each direction beyond which David's repulsive force dominates, repelling the girl from this "dipole". Conversely, at smaller distances, there is a non-zero neighborhood of Peter where his attractive force prevails. The region of search is therefore finite and non-zero. It is worth noting the symmetry of the problem with respect to the axis of symmetry passing through David and Peter. Hence, we will look for a solution in only one half-plane.

First, we will try to solve the problem analytically. We are interested in the angle between the resultant force acting on the girl and the line joining the girl and Peter. It is natural to describe the problem using polar coordinates $(r_\mathrm{P}, \theta_\mathrm{P})$, placing Peter at the origin. The axis from which we measure the angle θ_P will be directed towards David. Peter's attractive force then has the simple form:

$$\mathbf{F}_\mathrm{P} = -\frac{k_\mathrm{P}}{r_\mathrm{P}^3}\hat{\mathbf{e}}_{r_\mathrm{P}} \,.$$

Consider the triangle formed by Peter, David, and the girl. The square of the distance between David and the girl can be determined using the cosine theorem:

$$r_\mathrm{D}^2 = r_\mathrm{P}^2 + d^2 - 2dr_\mathrm{P}\cos\theta_\mathrm{P} \,.$$

Using the sine theorem, we can similarly determine the angle at the vertex where the girl is located:

$$\varphi = \arcsin\left(\frac{d}{r_\mathrm{D}}\sin\theta_\mathrm{P}\right) \,.$$

David's repulsive force can then be decomposed into the components

$$\mathbf{F}_\mathrm{D} = \frac{k_\mathrm{D}}{r_\mathrm{D}^2}\left(\cos\varphi\ \hat{\mathbf{e}}_{r_\mathrm{P}} + \sin\varphi\ \hat{\mathbf{e}}_{\theta_\mathrm{P}}\right) \,.$$

The solution to the problem is to find the region where the resultant force draws the girl closer to Peter, thus

$$(\mathbf{F}_\mathrm{P} + \mathbf{F}_\mathrm{D}) \cdot \hat{\mathbf{e}}_{r_\mathrm{P}} < 0 \,.$$

From the shape of the forces and the geometric sketch of the situation, we see that to find the boundary $b(\theta)$ we would have to solve the cubic equation for all $\theta \in (0, \pi)$. Even if we succeeded, we would still need to integrate the resulting form. Analytically, we get no further; we are left to solve the problem experimentally or through simulation. From a time and ethical point of view, we choose the latter.

One way to numerically determine the area we are looking for is to calculate the force field on the grid and determine the number of points at which the specified condition is satisfied. In this case, it is worth switching to Cartesian coordinates

$$x = r_\mathrm{P}\cos\theta_\mathrm{P} \quad \text{a} \quad y = r_\mathrm{P}\sin\theta_\mathrm{P} \,,$$

where for the distances of the girl from Peter and David

$$\begin{aligned} r_\mathrm{P}^2 &= x^2 + y^2 \,, \\ r_\mathrm{D}^2 &= (x - d)^2 + y^2 \,. \end{aligned}$$

The angles between the x-axis and the individual lines are

$$\theta_\mathrm{P} = \arctan \frac{y}{x} \,,$$
$$\theta_\mathrm{D} = \arctan \frac{y}{x-d} \,.$$

From the decomposition of the forces

$$\mathbf{F}_\mathrm{P} = -\frac{k_\mathrm{P}}{r_\mathrm{P}^3} \left(\cos\theta_\mathrm{P}\ \hat{\mathbf{e}}_x + \sin\theta_\mathrm{P}\ \hat{\mathbf{e}}_y\right) \,,$$
$$\mathbf{F}_\mathrm{D} = \frac{k_\mathrm{D}}{r_\mathrm{D}^2} \left(\cos\theta_\mathrm{D}\ \hat{\mathbf{e}}_x + \sin\theta_\mathrm{D}\ \hat{\mathbf{e}}_y\right) \,,$$

we get the first part of the simulation, namely a function that assigns to each point on the grid the force acting on the girl at that point. We see the result in Figure 6.

```
def F(x: float, y:float) -> (float, float):
        phi_P = atan2(y, x)
        phi_D = atan2(y, x - d)

                F_P = k_P / (x ** 2 + y ** 2) ** (3/2)
                F_D = k_D / ((x - d) ** 2 + y ** 2)

                F_x = F_D * cos(phi_D) - F_P * cos(phi_P)
                F_y = F_D * sin(phi_D) - F_P * sin(phi_P)

        return F_x, F_y
```

For a suitable choice of the grid, we go through all its points and determine at how many of them the condition is satisfied

$$\left(\mathbf{F}_\mathrm{P} + \mathbf{F}_\mathrm{D}\right) \cdot \left(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y\right) < 0 \,.$$

The latter just expresses differently the requirement that the resulting force has a positive component in the direction towards Peter, since $x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ represents the position vector from Peter to the girl.

```
N = 0
for i in range(grid_height):
        for j in range(grid_width):
                x = dx * (j - grid_width // 2) + d / 2 # origin in the middle of the grid
                y = dx * i
                try:
                        F_x, F_y = F(x, y)
                        if x * F_x + y * F_y < 0:
                                N += 1
                except ZeroDivisionError:
                        N += 1
```

Using this simulation with the result in Figure 7, we obtain the number of points N we are looking for, from which we determine the unknown area as

$$S \ = 2N(\mathrm{d}x)^2 \,.$$

The question remains as to how to choose a suitable grid shape. The most pronounced interaction with Peter will occur on the x-axis in the negative direction. If we determine the point at which the repulsive force of David starts to dominate, we get an estimate of the appropriate lattice size. In finding the appropriate size of the dx-step on the lattice, we can proceed by gradually reducing it until the resulting area varies beyond the desired precision.

From the graph in Figure 8, we see that for a relatively large step relative to the computational power, we already get a pretty good estimate of

$$S = 2.69\,\mathrm{m}^2\,.$$

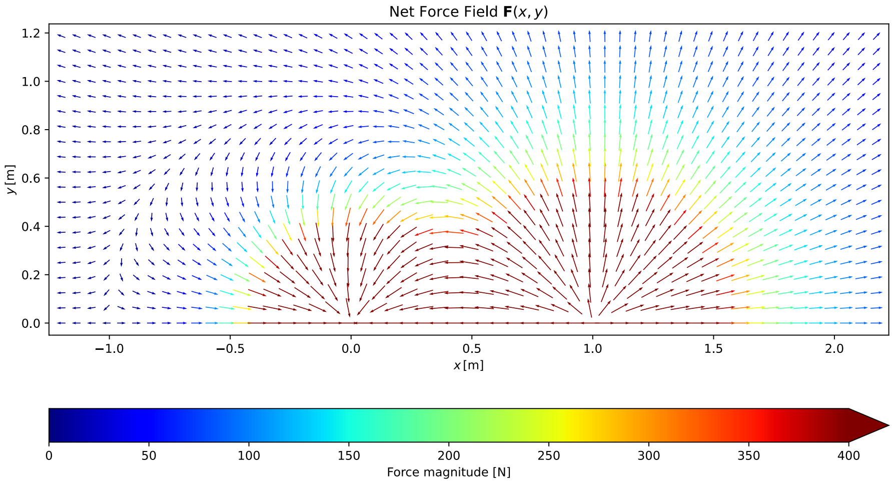


Figure 6: Result of the first simulation: the force field on the grid.

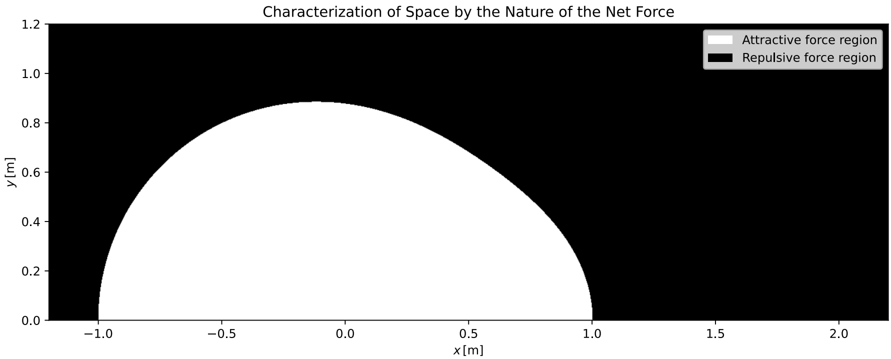


Figure 7: Result of the second simulation: areas with attracting and repulsive force.

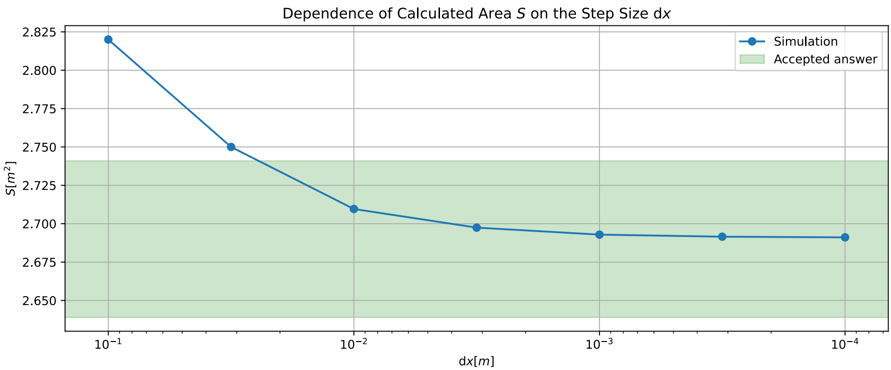


Figure 8: Graph of the dependence of the calculated area size on the chosen step size.

Problem 54 ... the fullest spoon

9 points

The inner volume of a spoon is an ellipsoid with semi-axes $a = 4.0\,\mathrm{cm}$*,* $b = 3.0\,\mathrm{cm}$ *and* $c = (2.0 + \sqrt{3})\,\mathrm{cm}$ *and height* $h = 5.0\,\mathrm{mm}$ *(see figure). Let us pour salt on it, which has a critical angle of repose* $\theta = 40^\circ$*. What is the maximum volume of salt we can carry on the spoon?*

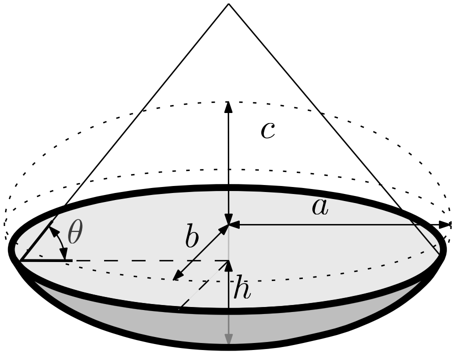


Figure 9: Schema of the spoon.

Jarda always puts just one sugar into his tea.

We will split the solution into two parts, where we will begin with finding the volume of the spoon and then focus on the volume of poured substance. Let us look at integration of volume of ellipsoid segment. The equation for ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,,$$

from which we can easily extract the equation for ellipse on section at height z

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}\,,$$

$$\frac{x^2}{a^2\left(1 - \frac{z^2}{c^2}\right)} + \frac{y^2}{b^2\left(1 - \frac{z^2}{c^2}\right)} = 1\,.$$

By using the area of ellipse $S = \pi ab$ with axes a and b we can express the volume of ellipsoid segment as:

$$\begin{aligned} V_1 &= \int_{c-h}^{c} \pi ab \left(1 - \frac{z^2}{c^2}\right) \mathrm{d}z = \pi ab \left(z - \frac{z^3}{3c^2}\right)_{c-h}^{c} = \pi ab \left[c - (c-h) - \frac{c^3 - (c-h)^3}{3c^2}\right]\,, \\ &= \pi ab \left(h - \frac{3c^2 h - 3ch^2 + h^3}{3c^2}\right) = \pi ab \left(\frac{3ch^2 - h^3}{3c^2}\right) = \pi ab \left(\frac{h^2}{c} - \frac{h^3}{3c^2}\right)\,, \\ &= \frac{\pi}{2}\left(5 - 2\sqrt{3}\right)\,\mathrm{cm}^3 \doteq 2.4\,\mathrm{cm}^3\,. \end{aligned}$$

Now we shall focus at the "hill" on the spoon. Maximal height of each point is found using the distance $s_{\min}$ to the nearest point of edge ellipse as

$$h_{\max} = s_{\min} \tan\theta \,.$$

Upper edge of the spoon is an ellipse with axes $a' = a\sqrt{1-(c-h)^2/c^2} = 2\,\mathrm{cm}$ and $b' = b\sqrt{1-(c-h)^2/c^2} = 1.5\,\mathrm{cm}$. If we tried to find the distance s of point x_0 and y_0 from the ellipse, we get expression

$$s^2 = (x-x_0)^2 + (y-y_0)^2 = (x-x_0)^2 + \left(b'\sqrt{1-\frac{x^2}{(a')^2}} - y_0\right)^2 \,,$$

whose derivative and its alignment to zero gives us the x coordinate of the point on said ellipse with the distance $s_{\min}$. However, the derivative yields a fourth degree equation, way beyond our ability to easily solve it.

Therefore, we will solve this numerically. We will divide the ellipse with dimensions a' and b' with square grid and for each point decide, whether it lies inside the ellipse. If it does, we find the smallest distance between it and the ellipse and add the element of volume $\Delta V = \Delta x \Delta y s_{\min} \tan\theta$. This results in volume $V_2 = 4.4\,\mathrm{cm}^3$ for ellipse with axes $a' = 2\,\mathrm{cm}$ and $b' = 1.5\,\mathrm{cm}$. The simulation code in language `Julia` is stated below.

```
a=2
b=1.5 # zoberiem stvrtku lyzice

Mi=601
Mj=801 # it is setup, that dx=dy, rest of the code depends on that
pole=zeros(Mi,Mj)

dx=a/(Mj-1)
dy=b/(Mi-1)

for i=1:Mi
y=(i-1)*dy
y2b2=y^2/b^2
for j=1:Mj
x=(j-1)*dx
x2a2=x^2/a^2
pole[i,j]=1-y2b2-x2a2 # this is positive when the pole is inside of spoon, otherwise is
    negative
end
end

hjs=zeros(Int64,Mi)
for i=1:Mi
j=1
while pole[i,j]>0 # for evry i find biggest j that is from spoon
j+=1
end
hjs[i]=j
end


for i=1:Mi
# print(".")
for j=1:Mj
if pole[i,j]>0 # lookup for closest point from rim of the spoon
minvz=Mj
```

```
di=0
while hjs[i+di]>j # looking up only in queter of plane
vz=sqrt(di^2+(hjs[i+di]-j)^2 )
if vz<minvz
minvz=vz
end
di+=1
end
pole[i,j]=minvz
else
pole[i,j]=0 # outside of spoon "zero height"
end
end
end
sum(pole)*dx*dy*dx*tan(40/180*pi)*4 # dx^3 the volume of one little square is the side
     length * tan(40) * 4 because we were looking at 1/4 of the spoon
```

The result is the sum of the two volumes

$$V_1 + V_2 = 2.4\,\mathrm{cm}^3 + 4.4\,\mathrm{cm}^3 = 6.8\,\mathrm{cm}^3\,.$$

Problem 55 ... Faraday's collector — 8 points

Consider a conductive circular target of radius $r = 5.00\,\mathrm{mm}$, onto which positively charged ions impinge perpendicularly with areal flux $j = 2.00\cdot 10^8\,\mathrm{cm}^{-2}{\cdot}\mathrm{s}^{-1}$ and charge magnitude equal to the elementary charge. However, upon impact of one ion, on average $\sigma = 0.230$ secondary electrons are released, which leave the surface with a distribution described by the probability density $f = (1/\pi)\sin\alpha$ per unit solid angle with respect to spherical coordinates centered at the point of impact. The angle α here is measured from the surface of the disk toward the normal. What current do we measure in the grounded linear conductor connected to the disk if we also make a fence around the target of height $h = 2.00\,\mathrm{mm}$, which is conductively connected to the disk and is able to capture these electrons? Assume that the motions of the particles are not influenced by any electric field.

First, we compute the total current that would be caused by the impact of the positively charged ions if no secondary electrons were released. From their areal flux and the dimensions of the target, we simply obtain

$$I_+ = qj\pi r^2\,,$$

where q is the magnitude of the elementary charge. However, to this current, it is necessary to add the effect of the secondary electrons, because the target loses the negative charge of all electrons that do not impinge on the fence; therefore, the resulting current will be higher.

It remains to calculate how many secondary electrons per second miss the fence.

The secondary electrons are emitted from every point of the surface into the hemisphere above the plane of impact. The probability that electrons are emitted at an angle α—The angle α is the inclination with respect to the plane of the target—in the range$(0; \pi/2)$ and in the direction φ is described by the probability density $f = (1/\pi)\sin\alpha$. Thus, the probability that they are emitted into the solid angle element $\mathrm{d}\Omega = \cos\alpha\,\mathrm{d}\varphi\,\mathrm{d}\alpha$ is equal to

$$\mathrm{d}P = \frac{1}{\pi}\cos\alpha\sin\alpha\,\mathrm{d}\varphi\,\mathrm{d}\alpha\,.$$

From every point of the target with area $\mathrm{d}S$, on average $\sigma j \,\mathrm{d}S$ electrons are emitted per second. Thus, for further considerations, we select an infinitesimal area element with area $\mathrm{d}S$ at a distance R from the center of the target and express how many electrons emitted from this region miss the fence around the target. First, consider only the number of electrons emitted in a narrow strip bounded by the angle $\mathrm{d}\varphi$ in the direction φ, measured from the ray originating in the considered infinitesimal area and passing through the center of the target. Let us try to express the minimum angle α below which they can be emitted to just miss the fence (the angle to the upper edge of the fence). Using the law of cosines, we express the dependence of the distance d of the fence in the given direction from the considered area element on the angle φ as

$$r^2 = R^2 + d^2 - 2Rd\cos\varphi\,,$$

from which solving the quadratic equation yields

$$d = R\cos\varphi + \sqrt{r^2 - R^2\left(1-\cos^2\varphi\right)}\,.$$

Here, we consider only the positive solution; the negative has no physical meaning. The minimum angle below which electrons can leave the surface in the given direction without hitting the fence is therefore equal to

$$\alpha_{\mathrm{min}} = \arctan\frac{h}{d}\,.$$

To now obtain the total number of electrons that miss the fence in this direction, we must integrate the probability density over the range $(\alpha_{\mathrm{min}};\, \pi/2)$ and multiply the result by the total number of electrons emitted from this area element. Next, we perform the integration over all directions, i.e., from 0 to 2π according to φ. This yields the total contribution from this area element. Finally, we must sum the contributions from all such area elements over the entire target. We can consider that all infinitesimal area elements at the same distance have the same contribution, so for $\mathrm{d}S$ we can substitute $2\pi R\,\mathrm{d}R$, which is the area of the ring at constant distance from the center of the target. We integrate according to R. The resulting integral that we need to compute to determine the secondary current due to the loss of secondary electrons has the form

$$I_{\mathrm{s}} = \int_0^r \int_0^{2\pi} \int_{\alpha_{\mathrm{min}}}^{\frac{\pi}{2}} e\sigma j 2\pi R \frac{1}{\pi}\cos\alpha\sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\varphi\,\mathrm{d}R\,,$$

where we have also multiplied the number of lost electrons by their charge to determine the current. We solve the first integral by the substitution $t := \sin\alpha$, which after solution gives

$$I_{\mathrm{s}} = \int_0^r \int_0^{2\pi} e\sigma j 2\pi R \frac{1}{2\pi}\left(1-\sin^2\alpha_{\mathrm{min}}\right)\mathrm{d}\varphi\,\mathrm{d}R\,.$$

Next, we use the trigonometric identity

$$\sin^2\alpha = \frac{\tan^2\alpha}{1+\tan^2\alpha}\,,$$

thanks to which, we can rewrite the previous integral as

$$I_{\mathrm{s}} = \int_0^r \int_0^{2\pi} e\sigma \frac{2\pi}{2\pi} jR \left(1 - \frac{h^2}{h^2 + \left(R\cos\varphi + \sqrt{r^2 - R^2\left(\sin^2\varphi\right)}\right)^2}\right)\mathrm{d}\varphi\,\mathrm{d}R\,.$$

We solve this integral numerically using the `dblquad` function from the `scipy.integrate` library in the Python programming language. The resulting value is $I_\mathrm{s} = 3.89 \cdot 10^{-12}$ A. The total current I is then given by the sum of the currents caused by the impact of positive ions and by the loss of secondary electrons

$$I = I_+ + I_\mathrm{s} = qj\pi r^2 + I_\mathrm{s} = 2.91 \cdot 10^{-11}\ \mathrm{A}\,.$$

The program used for the numerical computation of the integral is attached below.

```
from scipy.integrate import dblquad

r = 0.5
h = 0.2
j = 2.0e8
sigma = 0.23
e = 1.602176634e-19

def integrand(phi, R):
arg = r**2 - R**2 * np.sin(phi)**2
sqrt_term = np.sqrt(arg)
A = R * np.cos(phi) + sqrt_term
return R * (1 - (h**2) / (h**2 + A**2))

result, error = dblquad(integrand, 0, r, lambda R: 0, lambda R: 2*np.pi)
I_s = e * sigma * j * result

print(f"I_s = I_s:.3e A")
print(f"Odhad chyby integrace: e*sigma*j*error:.2e")
```

Problem 56 ... escape across a frozen dam — 9 points

A hiker was trying to flee from a hungry wolf in winter, and his path led past a lowered frozen dam. The animal was at his heels, so he had to plan his route as efficiently as possible. He was going to climb the concrete dam, where he would be safe. There were thick reeds all along the bank, but suddenly, he saw a clear area of ice. He ran onto the frozen surface at $u = 6.00\,\mathrm{m{\cdot}s^{-1}}$ in a direction parallel to the dam, from which he was $d = 150\,\mathrm{m}$ away. What is the shortest time to reach safety if the coefficient of friction between the ice and his boots is $f = 0.220$ and if he has to stop at the dam to climb it, which will take another $10.0\,\mathrm{s}$?

Jarda participated in the Physics Cup.

On a frozen surface, the tourist can move at most with acceleration $a = fg$, where g is the acceleration due to gravity. If he wanted to change his speed faster, his shoes would start to slip. But the resultant forces can act in any direction.

Let us define a Cartesian coordinate system where the tourist is at the point $(0, 0)$ when entering the lake, and his velocity is in the direction of the x axis, so $v_x = u$, while $v_y = 0$. Let's locate the dam at coordinates $y = d$. Velocity must be zero at the dam. We must choose the one that gets to the dam the fastest from all possible trajectories.

Examine its motion in space with velocity v_x, v_y. The initial velocities are $(u, 0)$ and the final velocities are $(0, 0)$. It is most advantageous for the tourist to move with as much acceleration as possible, so let $\left(\frac{\mathrm{d}v_x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}v_y}{\mathrm{d}t}\right)^2 = f^2g^2$. Moreover, let v_y be a function of v_x here, then the velocity element in this diagram will be $\sqrt{1+\left(\frac{\mathrm{d}v_y}{\mathrm{d}v_x}\right)^2}\,\mathrm{d}v_x$. Using the previous equation, we get

$$-\sqrt{1+(v_y')^2}\,\mathrm{d}v_x = fg\,\mathrm{d}t\,,$$

where the comma above v_y denotes the derivative by v_x. The negative sign here means that v_x decreases with increasing time. If it were increasing, it would have to go to zero anyway, so the tourist would lose time.

We can now express the time spent moving on the ice as

$$T = -\frac{1}{fg}\int_u^0 \sqrt{1+(v_y')^2}\,\mathrm{d}v_x\,.$$

But we still need to find the v_y' function. However, we know the coordinates of the dam from the input. We express the path to it using the equation

$$d = \int_0^T v_y\,\mathrm{d}t = -\frac{1}{fg}\int_u^0 v_y\sqrt{1+(v_y')^2}\,\mathrm{d}v_x\,,$$

where we substituted $\mathrm{d}t$ from the equation above.

Now, we could use the calculus of variations and solve the Euler-Lagrange equations. But looking at both integrals carefully may remind us of another physics problem – the chain problem. When we hang both ends of a string to the same height in a gravitational field, we are looking for the shape of the string. We can find a solution by minimizing the potential energy, provided we know the length of the string we maintain. Here, the integral for T is equivalent to the condition of conservation of the string length, while the integral for d is equivalent to minimizing the energy by finding the position of the center of gravity.

We can use its well-known solution since integrals are equivalent to finding a chain rule. The function v_y thus takes the form

$$v_y = A\ \cosh\frac{v_x - C}{A} + B\,,$$

where A and B are constants determined by boundary conditions.

The first such condition is passing through the point $(0,0)$, which corresponds to stopping at the dam. From here, $B = -A\cosh C/A$. Then $A\cosh((u-C)/A) - A\cosh(C/A) = 0$, which in turn determines with which velocity components the tourist ran onto the ice surface. This corresponds to two possible solutions, either $u - C = -C$, where $u = 0$, which is not true according to the problem, or $u - C = C$, which leads to $C = u/2$.

Substituting y in the last of the conditions, we get

$$-\sqrt{1+(v_y')^2}\,\mathrm{d}v_x = -\sqrt{1+\left(\sinh\frac{2v_x-u}{2A}\right)^2}\,\mathrm{d}v_x = -\cosh\frac{2v_x-u}{2A}\,\mathrm{d}v_x = fg\,\mathrm{d}t\,.$$

Then, we have

$$\begin{aligned} d &= \int_0^T v_y\,\mathrm{d}t = -\frac{1}{fg}\int_u^0 \left(A\cosh\frac{2v_x-u}{2A} - A\cosh\frac{u}{2A}\right)\cosh\frac{2v_x-u}{2A}\,\mathrm{d}v_x \\ &= \frac{A}{2fg}\left(A\sinh\frac{u}{A} - u\right)\,, \end{aligned}$$

from where we express $A = 0.783\,0\,\mathrm{m{\cdot}s^{-1}}$.

The desired time of movement on the ice is

$$T = -\frac{1}{fg}\int_u^0 \cosh\frac{2v_x - u}{2A}\,\mathrm{d}v_x = \frac{2A}{fg}\sinh\frac{u}{2A}\,.$$

After adding A, add ten more seconds, and you have the result $T_f = T + 10.0\,\mathrm{s} = 26.7\,\mathrm{s}$.

Problem B.1 ... mighty power bank in action — 3 points

We are buying a $30\,000\,\mathrm{mA{\cdot}h}$ *power bank for* $30\,\mathrm{USD}$*. What is the minimum fraction of the total price represented by the lithium it contains, given that the market price of lithium is* $11.4\,\mathrm{USD{\cdot}kg^{-1}}$*? The power bank operates using Li-ion battery technology.*

Jarda was frightened by the increase in the price of lithium.

The capacity of a battery or power bank indicates the total charge it is capable of transferring. This value can be converted to coulombs, the SI unit of charge, i.e. $Q = 30\,000\,\mathrm{mA{\cdot}h} = 108\,\mathrm{kC}$, where we multiplied the current by the duration of one hour in seconds.

In a Li-ion battery, lithium ions are transferred between the electrodes. Lithium is a monovalent element, so it transfers as singly ionized Li^+ in the electrolyte. For each lithium atom transferred, the elementary charge of one electron e is involved. By dividing the total charge Q by this value, we get the number of lithium atoms needed for the power bank to have the specified capacity.

Since working with such large numbers, like the number of atoms, is impractical, we will continue with calculations in moles. One mole is equivalent to $N_\mathrm{A} \doteq 6.022\cdot 10^{23}$ atoms or particles, where N_A is Avogadro's constant. The number of moles of lithium in the power bank is then at least

$$n = \frac{Q}{eN_\mathrm{A}} = \frac{Q}{F} \doteq 1.12\,\mathrm{mol}\,,$$

where we introduced Faraday's constant $F = eN_\mathrm{A}$.

Next, we need to convert the amount of substance to mass to determine the cost of the lithium used. For this, we need the molar mass of lithium, $M_\mathrm{Li} \doteq 6.941\,\mathrm{g{\cdot}mol^{-1}}$, which can be easily found online. The power bank therefore contains at least

$$m = nM_\mathrm{Li} \doteq 7.77\,\mathrm{g}$$

of lithium, which costs

$$P = mp \doteq 0.0886\,\mathrm{USD}\,,$$

for a price per kilogram of $p = 11.4\,\mathrm{USD{\cdot}kg^{-1}}$; which constitutes

$$\frac{0.0886\,\mathrm{USD}}{30\,\mathrm{USD}} \doteq 0.30\,\%$$

of the total price of the power bank.

Problem B.2 ... heavy charging 3 points

Manufacturers may consider lithium still quite expensive, and so they begin producing power banks that use sodium. By how much would the mass of a power bank increase to have the same capacity of 30 000 mA·h*? All other components of the device remain unchanged.*

Jarda wondered whether his weary back would wear out.

From the previous part, we already know the minimum mass of lithium in the power bank, since we know its amount of substance n. Sodium atoms are heavier than lithium atoms, but we must determine how many there will be. Sodium, like lithium, is an element of the first group of the periodic table, so it would also appear in the battery as singly ionized. Therefore, for the same charge Q, we need the same number of moles n of sodium atoms as lithium atoms.

The only difference lies in the molar mass. The total mass thus increases by

$$\Delta m = n\left(M_{\mathrm{Na}} - M_{\mathrm{Li}}\right) = \frac{Q}{F}\left(M_{\mathrm{Na}} - M_{\mathrm{Li}}\right) \doteq 18\,\mathrm{g}\,,$$

where we have, in turn, looked up the molar mass of sodium $M_{\mathrm{Na}} \doteq 22.99\,\mathrm{g{\cdot}mol^{-1}}$.

Problem B.3 ... I don't have the capacity for that 4 points

A 30 000 mA·h *power bank is connected to a DC source of* 5.0 V *through a* 1.5 m *cable. It charges from* 0 *to* 50% *capacity in* 6.5 h. *Determine the cable's resistance per unit length (measured in* $\Omega{\cdot}\mathrm{m}^{-1}$*), assuming that the charging circuitry of the power bank requires a constant voltage of* 4.1 V*, independent of the current.*

Jarda's batteries ran out after a demanding week.

The power bank operates via electrochemical processes that involve the transport of ions between the electrodes. According to the problem statement, a voltage $U_{\mathrm{p}} = 4.1\,\mathrm{V}$ is required for these processes to occur during charging. However, the supply provides $U_{\mathrm{s}} = 5.0\,\mathrm{V}$. The excess voltage is therefore dropped across the ohmic resistance of the supply cable.

The voltage across the full length of the cable (there and back) is

$$U_{\mathrm{s}} - U_{\mathrm{p}} = R_1 I_1\,,$$

where R_1 denotes the total resistance of the cable and I_1 the current through it. This current may be obtained from the charging time and the transferred charge:

$$I_1 = \frac{0.5Q}{t} = \frac{0.5 \cdot 30\,\mathrm{A{\cdot}h}}{6.5\,\mathrm{h}} = 2.3\,\mathrm{A}\,,$$

the factor 0.5 representing fifty percent of the power bank's total capacity.

The total cable resistance is therefore

$$R_1 = \frac{U_{\mathrm{s}} - U_{\mathrm{p}}}{I_1} \doteq 0.39\,\Omega\,.$$

Per unit length of the USB cable this corresponds to

$$\lambda = \frac{R_1}{d} \doteq 0.26\,\Omega{\cdot}\mathrm{m}^{-1}\,,$$

where d is the length of the cable. Although in reality the charge must traverse the cable twice (there and back), the problem asks for the length-specific resistance of the entire USB cable. The value obtained is relatively large because the assumption that the voltage required by the power bank is constant is not entirely physical: as the current increases a higher supply voltage is generally required.

Problem B.4 ... (in)efficient powerbank — 4 points

The electrochemical reactions in our $30\,000\,\mathrm{mA{\cdot}h}$ *power bank provide a voltage of* $3.80\,\mathrm{V}$*. However, the output generates a USB voltage of* $5.00\,\mathrm{V}$*. If the energy loss during this conversion is* 15.0%*, what is the effective capacity of the power bank for charging a mobile phone?*

Jarda did not recharge.

The capacity of our power bank is $Q = 30.0\,\mathrm{A{\cdot}h}$, representing the total stored charge. Multiplying by the electrochemical reaction voltage $U_\mathrm{p} = 3.80\,\mathrm{V}$ gives the total stored energy,

$$E = QU_\mathrm{p}\,.$$

This energy can be delivered at the output voltage $U_\mathrm{o} = 5.00\,\mathrm{V}$. Since the output voltage is higher, the effective charge that can be supplied is smaller. In addition, part of the energy is dissipated during the voltage conversion. Thus, the effective capacity is

$$Q_\mathrm{ef} = 0.850\frac{U_\mathrm{p}}{U_\mathrm{o}}Q \doteq 1.94 \cdot 10^4\,\mathrm{mA{\cdot}h}\,,$$

where 0.850 is the efficiency of the step-up voltage conversion.

Problem P.1 ... knocking the opposing boule away — 4 points

Jirka is playing pétanque with Jarda. Unfortunately, Jarda is currently winning, meaning that one of his boules is very close to the small wooden target ball (the so-called jack). The only way Jirka can turn the game in his favor is to knock Jarda's boule away so that it ends up as far from the jack as possible. Therefore, he throws his boule so that it hits Jarda's boule with a horizontal speed of $40\,\mathrm{cm{\cdot}s^{-1}}$*. How far will Jarda's boule roll if, after the collision, it is acted upon by a constant resistive force of magnitude* $F = \alpha mg$*, where* $m = 700\,\mathrm{g}$ *is the mass of the boule,* g *is the gravitational acceleration, and* $\alpha = 0.010$*? Both boules are identical, the collision is perfectly elastic, and after the collision the boules move along the same straight line as Jirka's boule before impact (so the collision can be treated as a one-dimensional elastic collision of point masses).*

Jirka never played pétanque with Jirka.

We divide the solution conceptually into two parts. First, we compute the result of the collision, and then from Jarda's boule's post-collision velocity we determine how far it will roll.

For an elastic collision we must satisfy conservation of both momentum and kinetic energy. In our case, the boules have equal mass, so one can straightforwardly infer that after the collision Jirka's boule comes to rest and transfers all its momentum to Jarda's boule (it is easy to verify that this outcome indeed satisfies both conservation laws). To obtain this result by calculation one would solve the system of two equations given by the conservation laws. In general the momentum conservation equation reads

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \,,$$

where m_1, m_2 are the masses, v_1, v_2 the pre-collision velocities and v_1', v_2' the post-collision velocities (the collision is one-dimensional, so the sign encodes direction). Likewise, energy conservation gives

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \,.$$

Substituting equal masses and $v_2 = 0$ into these equations yields, after elementary algebra, that the second boule's post-collision speed equals the first boule's pre-collision speed.

We also mention an alternative method using the center-of-mass frame, which will be particularly useful in the subsequent problems of this hurry-up series. For equal masses the center of mass lies midway between the boules. Denoting by v the speed of Jirka's boule before the collision, the center-of-mass speed relative to the ground is $v/2$. By momentum conservation the center-of-mass speed is unchanged by the collision. Thus, if we compute the collision outcome in the center-of-mass frame, we can then transform back to the laboratory frame.

In the center-of-mass frame the boules approach each other with speeds $v/2$. By symmetry, they separate after the collision with equal speeds of magnitude v'. Conservation of kinetic energy then implies $v' = v/2$. Transforming back to the original frame yields that Jirka's boule is at rest and Jarda's boule moves with speed v.

For the second part, Jarda's boule is decelerated by a constant resistive force of magnitude $\alpha m g$. Hence its acceleration (deceleration) has magnitude

$$a = \alpha g \,.$$

The distance covered under this uniform deceleration is

$$x = \frac{1}{2} a t^2 \,,$$

where $t = \frac{v}{a}$ is the stopping time. Substituting gives

$$x = \frac{v^2}{2\alpha g} \doteq 82\,\mathrm{cm} \,.$$

Problem P.2 ... slowing our ball down 5 points

After Jirka's brilliant throw in the ongoing pétanque match, Jarda lost his advantage. However, it is now his turn, and he is devising a risky plan. One of Jirka's boules is located just $5.0\,\mathrm{cm}$ *from the jack. Jarda knows that the collision of the boules is not perfectly elastic, so his boule will continue moving at a low speed even after a collision. Jarda intends to throw his boule in such a way that it will deflect Jirka's boule and still roll to the jack (Jarda believes that Jirka's deflected boule will not knock the jack away). What must be the speed of Jarda's boule just before the collision in order for it to travel the distance of* $5.0\,\mathrm{cm}$ *after the collision? Consider the collision as a collision of identical point masses in a straight line. A resistive force acts on the rolling boule with magnitude* $F = \alpha m g$*, where* $\alpha = 0.010$*, as in the previous problem. Jarda* *measured before the game that two identical pétanque boules always lose kinetic energy* $\Delta E = 0.25\,\Delta E_{\max}$ *during a straight-line collision, where* $\Delta E_{\max}$ *is the maximum amount of kinetic energy that the identical boules could lose in a collision without violating the law of conservation of momentum.* *If Jirka were to play pétanque against Jirka, he would surely win.*

The problem is stated in a somewhat unusual way. It contains a claim about the loss of kinetic energy in a general collision. Since we are not required to prove this claim, we may use it to solve the problem in a clever way. We only need to realize that the collision may be observed from any inertial reference frame, and the claim about kinetic energy remains valid in this new frame as well.[8]

Which frame should we choose? It is advantageous to choose the center-of-mass frame, because in this frame we immediately know the maximum possible energy loss $\Delta E_{\max}$. In the center-of-mass frame it is possible for the boules to come to a complete stop, and therefore $\Delta E_{\max} = E_{\mathrm{k}}$.

Let the unknown velocity of Jarda's boule be v. In the center-of-mass frame both boules move toward each other with speed $\frac{v}{2}$. The total kinetic energy is therefore

$$E_{k,0} = \frac{1}{4}mv^2\,.$$

After the collision they retain an energy of $(1-k)E_{\mathrm{k},0}$ where $k = 0.25$. Because the boules have the same mass and momentum is conserved, they share the remaining kinetic energy equally. The energy of one boule is thus

$$E_k = \frac{1-k}{8}mv^2\,,$$

from which we easily obtain the speed $u' = \frac{v\sqrt{1-k}}{2}$. This is the speed of Jarda's boule in the center-of-mass frame. Transforming back into the rest frame, we get

$$v' = \frac{v}{2}(1-\sqrt{1-k})\,.$$

We now use the relation for the distance traveled in motion with constant deceleration, which we derived in the previous task in this hurry-up series:

$$x = \frac{v'^2}{2\alpha g}\,,$$

where $x = 5\,\mathrm{cm}$. Finally, we solve this equation for the unknown initial velocity of Jarda's boule:

$$v = \frac{\sqrt{8\alpha g x}}{1-\sqrt{1-k}} = 1.5\,\mathrm{m{\cdot}s^{-1}}\,.$$

Note

For interest, we add a comment and the assumptions behind the claim given in the problem statement: that two boules of masses m_1 and m_2 always lose the same fraction of the maximum possible energy loss in any collision.

[8] Of course, we could solve the problem directly in the original frame, but that would lead to more complicated equations.

First, recall that the kinetic energy of a system of point masses may be decomposed into two parts:

$$E_k = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \frac{1}{2} M v_T^2 + \sum_{i=1}^{N} \frac{1}{2} m_i (v_i - v_T)^2 \,.$$

This relation holds in any number of dimensions (the velocities may be vectors). It is proved simply by substituting the total mass $M = \sum_{i=1}^{N} m_i$ and the center-of-mass velocity $v_\mathrm{T} = \frac{1}{M}\sum_{i=1}^{N} m_i v_i$. This decomposition is useful because the first term corresponds exactly to the kinetic energy that the masses cannot lose. Therefore,

$$\Delta E_{\max} = \sum_{i=1}^{N} \frac{1}{2} m_i (v_i - v_T)^2 \,.$$

Assume that the point masses have initial velocities u_1 andvu_2 such that the total momentum is zero (i.e., these are velocities in the center-of-mass frame). After the collision, these velocities must change to Cu_1 and Cu_2, where $0 \leq C \leq 1$. The energy loss is then

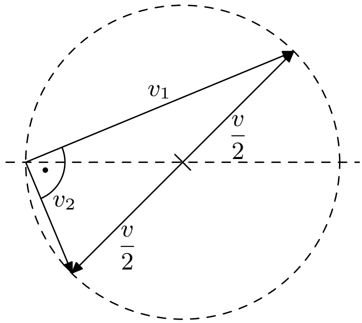
$$\Delta E = (1 - C^2)\Delta E_{\max} = \frac{1}{2}(m_1 u_1^2 + m_2 u_2^2)(1 - C^2) \,.$$

Now switch to a frame moving with velocity v_T relative to the center-of-mass frame. The velocities before the collision are then $u_1 - v_\mathrm{T}$ and $u_2 + v_\mathrm{T}$, and after the collision they must be (by the principle of relativity) $-Cu_1 - v_\mathrm{T}$ and $-Cu_2 + v_\mathrm{T}$. Expanding the kinetic energy using the previous decomposition, we again find that the energy loss is $\Delta E = (1 - C^2)\Delta E_{\max}$. Thus we have proved the claim for initial velocities $v_1 = u_1 - v_\mathrm{T}$ and $v_2 = u_2 + v_\mathrm{T}$, where $\tilde{v}$ is arbitrary; that is, we have proved it for any fixed velocity difference. To prove the claim for truly arbitrary velocities, we must add the assumption that the coefficient C does not depend on the velocity difference. That is, if velocities u_1, u_2 relative to the center of mass change to Cu_1, Cu_2, then velocities αu_1, αu_2 change to $C\alpha u_1$, $C\alpha u_2$ for any α. Under this assumption we again obtain $\Delta E = (1 - C^2)\Delta E_{\max}$, and now for velocities $v_1 = \alpha u_1 - v_\mathrm{T}$ and $v_2 = \alpha u_2 + v_\mathrm{T}$. It is then easy to see that this covers all possible pairs of velocities.

Problem P.3 ... scattered balls — 3 points

Tension between Jirka and Jarda is rising. In their ongoing game of pétanque, there are only a few throws left, and Jarda has taken the lead thanks to his trick with a partially elastic collision. For Jirka to make up the deficit, he must likewise exploit more complex physical behavior in collisions of the boules. He therefore throws his next boule so that it does not collide with Jarda's boule along a straight line, but "in a plane". What is the maximum angle that the lines of the boules' trajectories can form after the collision? In other words, what is the maximum angle between the vector lines of the boules' velocities after the collision, if we model it as a collision of identical point masses in the plane, assuming only conservation of energy and momentum? Only consider situations in which both boules have nonzero velocities after the collision. *Jirka promised that the solution would be much nicer than its statement.*

We present two approaches to the solution. The first is geometric and utilizes properties of the center-of-mass frame. The second is more mathematical and is likely more general (in the sense of being applicable to similar problems with more complicated assumptions). Its disadvantage, however, is that it does not provide as good a physical intuition as the first approach.



We begin with the geometric method. We switch to the center-of-mass frame. This frame moves at velocity $v/2$ relative to the laboratory frame, so in it both boules approach each other with speed $v/2$. The outcome of such an elastic collision is well known: the boules fly apart again with speed $v/2$. However, this is a collision in a plane, so the line along which the boules move after the collision can make an arbitrary angle with their original direction of motion.

It remains to return to the laboratory frame by adding the velocity of the center of mass, $v/2$. We do this graphically, as shown in the figure. Now comes the trick: we realize that for any direction of the velocities in the center-of-mass frame after the collision, the tips of these velocity vectors lie on a circle (see the figure). This circle is Thales' circle, and therefore the velocities in the laboratory frame always form an angle of $90\,^\circ$, regardless of the specific outcome of the collision in the center-of-mass frame.

Now, the second approach: Let the mass of both boules be m, the initial velocity of Jirka's boule $\mathbf{v}$, and the velocities after the collision $\mathbf{v}_1'$ (Jirka's boule) and $\mathbf{v}_2'$ (Jarda's boule).

We write the conservation of momentum formula in vector form

$$m\mathbf{v} = m\mathbf{v}_1' + m\mathbf{v}_2'$$

which, after dividing by the mass, yields

$$\mathbf{v} = \mathbf{v}_1' + \mathbf{v}_2'.$$

Next, the conservation of energy can be formulated as

$$\frac{1}{2}mv^2 = \frac{1}{2}m{v_1'}^2 + \frac{1}{2}m{v_2'}^2$$

and after dividing by the mass

$$v^2 = {v_1'}^2 + {v_2'}^2\,.$$

Now comes the trick: we take the scalar product of the momentum equation with itself

$$v^2 = \|\mathbf{v}_1' + \mathbf{v}_2'\|^2 = {v_1'}^2 + {v_2'}^2 + 2\mathbf{v}_1' \cdot \mathbf{v}_2'$$

and substituting from energy conservation, we obtain

$$\mathbf{v}_1' \cdot \mathbf{v}_2' = 0\,.$$

The scalar product is zero, so for the angle α between the velocities, we have

$$\cos\alpha = 0\,.$$

The result is therefore equal to $90\,^\circ$, which corresponds with the previous method.

Problem P.4 ... scattered balls reloaded 5 points

Jirka planned his last throw in the ongoing game of pétanque so that after the collision, his and Jarda's boules would roll apart such that both trajectories form an angle of $45\,^\circ$ with the original direction of Jirka's boule (in the previous problem you convinced yourselves that this is indeed possible for the collision of two point masses in a plane). However, in his planning, Jirka forgot that pétanque boules are not point masses but rigid bodies, which, in the case of non-zero friction, affects the result.

Consider therefore the following situation: Jirka throws his boule so that its center moves along a straight line whose distance from the center of Jarda's boule is $\sqrt{2}R$, where $R = 4\,\mathrm{cm}$ is the radius of the boules. The speed of Jirka's boule immediately before the collision is $1\,\mathrm{m{\cdot}s^{-1}}$. The boules are identical, and the collision of the boules is elastic in the normal direction (with respect to the point of contact of both boules). In the tangential direction, large friction acts, so immediately after the collision, the relative speed of the contact points in the tangential direction is zero. At what angle do the boules roll apart under these assumptions?

Assume that Jirka's boule is not rotating about its vertical axis just before the collision; any possible rotation in other directions (e.g., due to rolling) will not affect the result. Effectively, the angular velocity of Jirka's boule just before the collision can be considered zero.

Jirka wanted to satisfy his craving for hard problems.

We switch to the center-of-mass frame, resulting in the situation as in the figure 10. The velocities of the balls are thus $v/2$, pointing toward each other. We decompose into the normal and tangential directions as

$$u_\parallel = u_\perp = \frac{v}{2\sqrt{2}}\,.$$

In the normal direction, it is an elastic collision; they roll apart again with velocity

$$u'_\perp = u_\perp = \frac{v}{2\sqrt{2}}\,,$$

which holds for both balls.

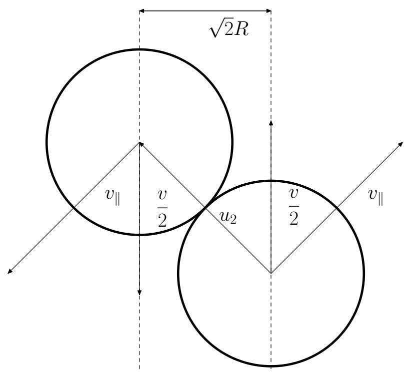


Figure 10

For the tangential direction, we proceed with the following reasoning: We know that after the collision, the relative velocity of the contact points is zero; in this direction, we must satisfy conservation of momentum and angular momentum (energy is not conserved).

The total momentum in the parallel direction before the collision is equal to zero (since we are in the center-of-mass frame). After the collision, we thus have the velocity

$$u'_{1\parallel} = u'_{2\parallel} = u'_\parallel\,.$$

We calculate the angular momentum with respect to the point of contact. The total angular momentum before the collision is non-zero and takes the form

$$L = 2L_1 = 2mu_{\parallel}R\,,$$

where $L_1 = mu_{\parallel}R$ is the angular momentum of one ball. This holds intuitively because the balls do not rotate. It is also possible to easily verify this by calculation

$$\mathbf{L} = \int \mathrm{d}m\,\mathbf{r}\times\mathbf{v}\,,$$

where only L_z is non-zero and the velocity has direction x, thus $v_x = \frac{v}{2}$

$$L_z = \frac{v}{2}\int \mathrm{d}m\,y = my_{\mathrm{T}}\frac{v}{2} = mu_{\parallel}R\,.$$

To express the angular momentum after the collision, we can use our knowledge about the zero velocity of the contact point. The angular velocity with respect to this point is equal to

$$\omega = \frac{u'_{\parallel}}{R}$$

and we express the moment of inertia from Steiner's theorem

$$J = mR^2 + \frac{2}{5}mR^2 = \frac{7}{5}mR^2\,.$$

To be precise, we should write that the sum of angular momenta is conserved, but due to $u'_{1\parallel} = u'_{2\parallel} = u'_{\parallel}$ this can also be written as $L_1 = L'_1$. It is therefore true that

$$mu_{\parallel}R = J\omega = \frac{7}{5}mRu'_{\parallel}\,,$$

which gives

$$u'_{\parallel} = \frac{5}{7}u_{\parallel} = \frac{5}{7}\frac{v}{2\sqrt{2}}\,.$$

To conclude, we solve the geometric part of the problem (fig. 11). We formulate the x and y components of the final velocity for both balls; for the first ball as

$$u'_{1x} = u'_{\parallel}\cos 45^\circ - u'_{\perp}\cos 45^\circ = -\frac{1}{\sqrt{2}}\frac{2}{7}u_{\parallel} = -\frac{1}{14}v\,,$$

and for the second ball

$$u'_{2x} = -u'_{\parallel}\cos 45^\circ + u'_{\perp}\cos 45^\circ = \frac{1}{\sqrt{2}}\frac{2}{7}u_{\parallel} = \frac{1}{14}v\,.$$

We can write the y axis component in the form

$$u'_{1y} = -u'_{2y} = \frac{1}{\sqrt{2}}\frac{12}{7}u_{\parallel} = \frac{6}{14}v\,.$$

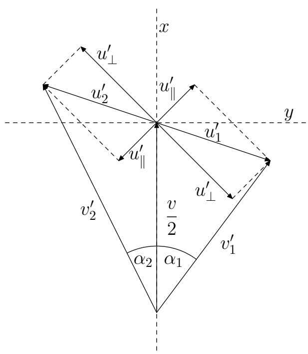


Figure 11

Now we return back to the laboratory frame by adding $\frac{v}{2}$ to the x component, obtaining the form

$$\mathbf{v}_1' = \frac{v}{14}(6,6) = \frac{3}{7}v(1,1)\,,$$

where the notation with parentheses symbolizes the vector components. Similarly, the velocity of the second ball

$$\mathbf{v}_2' = \frac{v}{14}(8,-6) = \frac{3}{7}v\left(\frac{4}{3},-1\right)\,.$$

It remains to calculate the angle that these two vectors form. Probably the simplest way is to calculate for each velocity the deflection angle from the original direction. For the first ball, we have $45\,^\circ$—note that this is the same result that we obtained in the case without considering friction—for the second ball $\alpha_2 = \arctan 3/4 \approx 37\,^\circ$.

The angle between the boules is equal to $82\,^\circ$.

Problem R.1 ... telescope illumination 4 points

Consider a section (spherical cap) of a concave spherical mirror with radius $R = 200\,\mathrm{cm}$ and radius of base of the cap $r = 150\,\mathrm{mm}$. In front of this mirror, place a circular ***opaque*** *screen of radius $\delta = 80\,\mathrm{mm}$ at a distance $d = 135\,\mathrm{cm}$ from the mirror's vertex, such that the center of the screen lies on the principal axis of the spherical mirror. Compute the ratio of the illuminated to the non-illuminated area on the screen's side facing the mirror. Assume that the light is incident parallel to the principal axis of the mirror. Feel free to use approximations valid for $r \ll R$ when solving this problem.*

Vlado was thinking about Cassegrain type telescope.

Consider a ray incident on the screen at a point where the tangent to the spherical surface of the mirror (at the point of incidence) forms an angle θ with the incoming ray. Owing to radial symmetry it suffices to analyse this situation in the plane that contains the centre of curvature of the mirror and both the incident and reflected rays. In that plane, the angle θ is also the angle between the incident ray and the line joining the centre of curvature of the mirror with the point of incidence. This line is the normal to the mirror at the point of reflection because it is perpendicular to the tangent plane of the mirror there.

For a ray at a distance ρ from the mirror axis, (see Figure 12) we obtain

$$\theta = \arcsin\left(\frac{\rho}{R}\right)\,.$$

We see that for $\rho < R$ the angle θ grows continuously with increasing ρ. Rays that reach the mirror for $\delta \leq r$ strike the mirror at radial distance ρ satisfying $\delta \leq \rho \leq r$, hence the corresponding angles θ for these rays assume all values from $\theta_{\min} = \arcsin\left(\delta/R\right) \approx 2.29\,^\circ$ up to $\theta_{\max} = \arcsin\left(r/R\right) \approx 4.30^\circ$.

The reflected ray then propagates along a straight line. This reflected line is the mirror image of the original incident line with respect to the radius through the point of incidence. The reflected ray therefore forms an angle 2θ with the incident ray. Under the assumption $r \ll R$, we may further assume that the reflected ray passes approximately through the mirror's focal point; thus it is sufficient to consider a half-line originating at the focal point at an angle 2θ with the principal axis of the mirror, since the incident ray was parallel to that axis.

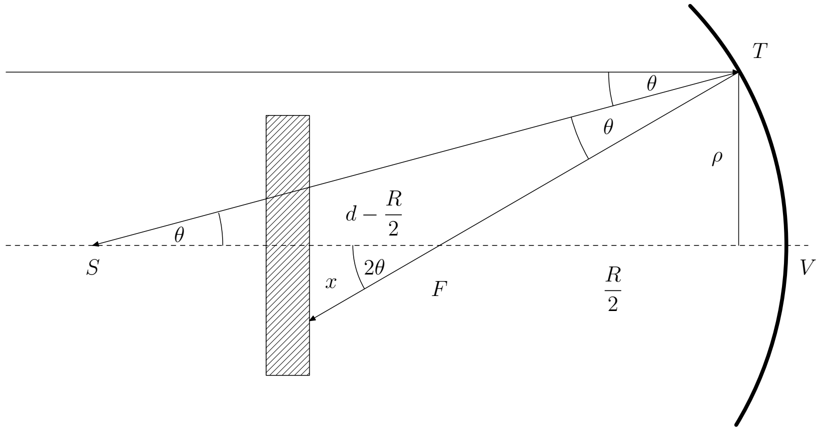


Figure 12: Reflection of a single ray from the mirror.

The focal point of a spherical mirror lies at a distance $R/2$ from the vertex; hence the focal point is at a distance $d - R/2$ from the centre of the screen. The distance x from the screen centre to the point where the reflected ray strikes the screen is given by the right triangle

$$x = \left(d - \frac{R}{2}\right) \tan\left(2\theta\right) \,.$$

Since in our case, θ attains a maximum value $\theta_{\max} = \arcsin\left(r/R\right) < \pi/2$, the value of x increases continuously with θ, and therefore x spans the interval from

$$x_{\min} = \left(d - \frac{R}{2}\right) \tan\left(2\theta_{\min}\right) \approx 28.1\,\mathrm{mm}$$

to

$$x_{\max} = \left(d - \frac{R}{2}\right) \tan\left(2\theta_{\max}\right) \approx 52.9\,\mathrm{mm}$$

for rays that strike the mirror.

The illuminated portion of the screen thus forms an annulus with inner radius $x_{\min}$ and outer radius $x_{\max}$, whose area is

$$S_{\mathrm{ill}} = \pi(x_{\max}^2 - x_{\min}^2)\,.$$

The total area of the side of the screen facing the mirror is $S_{\mathrm{total}} = \pi\delta^2$, hence the ratio of illuminated to non-illuminated area on that side is

$$\frac{S_{\mathrm{ill}}}{S_{\mathrm{total}} - S_{\mathrm{ill}}} = \frac{x_{\max}^2 - x_{\min}^2}{\delta^2 - x_{\max}^2 + x_{\min}^2} \doteq 0.460\,.$$

The inner and outer radii of the illuminated annulus could also be estimated, with slightly lower precision, by simple congruence of triangles. There we assume that the axial distance from the focal point to the point of reflection on the mirror is approximately $R/2$. For an incident ray at radial distance ρ from the optical axis one would then obtain

$$x \approx \rho \cdot \frac{d - \frac{R}{2}}{\frac{R}{2}} = \rho\left(\frac{2d}{R} - 1\right),$$

from which one finds $x_{\min} \approx 28.0\,\mathrm{mm}$ and $x_{\max} \approx 52.5\,\mathrm{mm}$. The corresponding ratio of illuminated to non-illuminated area would then be

$$\frac{S_{\mathrm{ill}}}{S_{\mathrm{total}} - S_{\mathrm{ill}}} = \frac{x_{\max}^2 - x_{\min}^2}{\delta^2 - x_{\max}^2 + x_{\min}^2} \doteq 0.445\,.$$

A direct simulation of the reflection of the extreme rays (for example in GeoGebra) shows that the true result is approximately 0.474. The second (simpler) solution therefore yields a somewhat less accurate value than the first. Nevertheless all three solutions were accepted, bearing in mind that even the first solution is not exact because the reflected rays do not, in reality, pass exactly through the focal point.

Problem R.2 ... shining on a teaspoon

4 points

Consider a section (spherical cap) of a concave spherical mirror with radius $R = 5.5\,\mathrm{cm}$ and radius of base of the cap $r = 4.0\,\mathrm{cm}$. A point source of light, which illuminates the reflective surface, is located at a distance $a = 7.0\,\mathrm{cm}$ from the vertex of the mirror. A screen is placed at a distance $d = 10\,\mathrm{cm}$ from the vertex of the mirror and is perpendicular to the principal axis. Calculate the radius of the image formed on the screen. Assume that the entire image can fit on the screen.

Dishes pile up on Vlado, yet he keeps playing with a teaspoon.

Unlike in the previous problem, in this task the mirror is no longer considered to be a small section of a spherical surface of radius R. Consequently, the approximations employed in the previous solution are not applicable here, because an exact result is required.

Consider a ray that emerges from the source at an angle φ with respect to the principal axis of the mirror. Denote by θ the angle between the incident ray and the line joining the centre of curvature of the mirror with the point at which the ray strikes the mirror.

From the triangle in Figure 13 these angles are related by the law of sines:

$$\frac{R}{\sin\varphi} = \frac{a - R}{\sin\theta}\,.$$

The reflected ray makes an angle 2θ with the incident ray. Since the incident ray forms an angle φ with the principal axis of the mirror and $a > R$, the reflected ray therefore makes an angle $\varphi + 2\theta$ with the axis.

The axial distance of the reflection point measured from the vertex equals $R - \sqrt{R^2 - \rho^2}$, where ρ is the radial distance of the reflection point from the axis. Hence the reflected ray travels, along the axis direction, a distance $d - R + \sqrt{R^2 - \rho^2}$ to reach the screen, at an angle $\varphi + 2\theta$ with respect to that axis. In the direction perpendicular to the axis it travels a distance

$$\left(d - R + \sqrt{R^2 - \rho^2}\right)\tan(\varphi + 2\theta)\,,$$

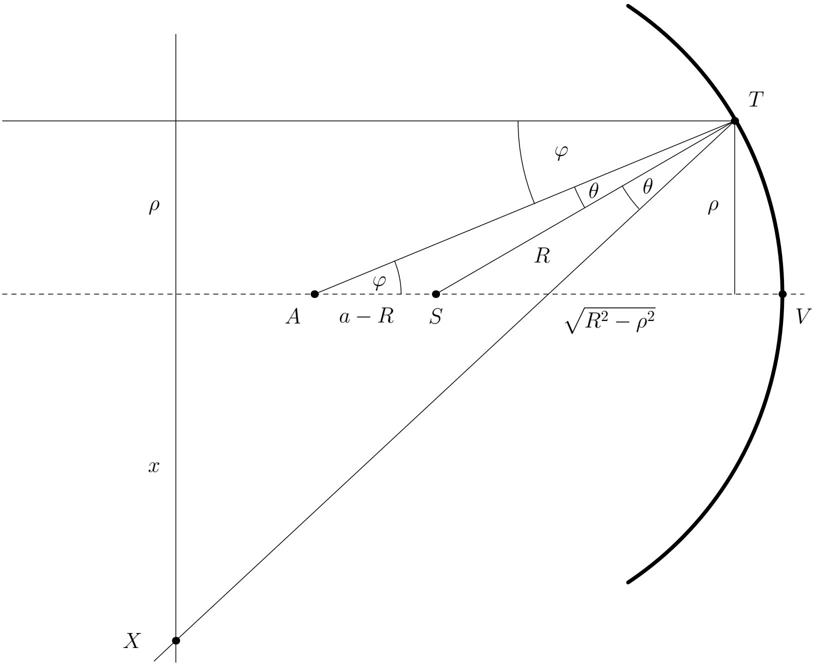


Figure 13: Reflection of a single ray from the mirror.

and the ray is reflected back toward the axis. Consequently it strikes the screen at a radial distance

$$x = \left(d - R + \sqrt{R^2 - \rho^2}\right)\tan(\varphi + 2\theta) - \rho$$

from the mirror axis, and thus from the centre of the screen.

It remains to determine the maximal value of x for rays that actually strike the mirror. Intuitively one expects that the maximum occurs for rays that strike precisely at the rim of the mirror, and in this case that expectation is indeed correct. To verify this more rigorously one may substitute the expression for θ in terms of φ into the formula for x, differentiate with respect to φ, and show that the derivative is always positive. Alternatively, plotting x as a function of φ (for instance in Desmos) provides a straightforward confirmation of this behavior.

Since the task is to determine the radius of the image on the screen, it suffices to consider the ray corresponding to the largest angle φ, namely the ray incident on the mirror rim. For that ray geometry gives

$$\varphi = \arctan\left(\frac{r}{a - R + \sqrt{R^2 - r^2}}\right) \approx 37.2^\circ\,,$$

and from the relation between φ and θ we compute

$$\theta = \arcsin\left(\frac{a - R}{R}\sin\varphi\right) \approx 9.48^\circ\,.$$

The image radius on the screen is therefore

$$x = \left(d - R + \sqrt{R^2 - r^2}\right)\tan(\varphi + 2\theta) - r \doteq 8.3\,\mathrm{cm}\,.$$

Problem R.3 ... reflector off the focal point 7 points

A car headlight consists of a parabolic mirror whose shape originated by rotating the parabola $z = v(r/r_0)^2$, $r < r_0$, where $r_0 = 7.0\,\mathrm{cm}$ and $v = 5.0\,\mathrm{cm}$. A negligibly small incoherent light source is located at its focus. Assume a screen is placed perpendicularly to the paraboloid axis at a distance $d = 1.0\,\mathrm{m}$ from the top edge of the reflector (plane $z = v$). However, the bulb is not exactly at the focus but on the paraboloid axis at a distance $\mathrm{d}h = -0.1\,\mathrm{cm}$ from it (closer to the paraboloid vertex). By how much does the radius of the bright circle at the center of the screen, where both direct light from the source and light reflected by the paraboloid fall, increase in this case?

Kuba could not see the road while driving.

It is sufficient to solve the problem in two dimensions because it is cylindrically symmetric. Let the parabola be described by the equation $z = ar^2$, where $a = v/r_0^2$, and let the end of the paraboloid be located at $A = [r_0, ar_0^2]$, and the source placed at the point $S = [0, h]$. It is enough to examine where the ray reflected from the edge of the paraboloid, i.e. at point A, will land. This will be the location of the boundary of the inner bright circle on the screen, since both rays reflected from the paraboloid and rays coming directly from the source fall inside it.

The ray incident at point A is described by the vector

$$\mathbf{v} = \overrightarrow{SA} = r_0\mathbf{e}_r + (ar_0^2 - h)\mathbf{e}_z\,.$$

This ray will reflect from the tangent to the parabola at point A, which we find by differentiating $z = ar^2$. The direction vector of the tangent has the form

$$\mathbf{t} = \mathbf{e}_r + 2ar_0\mathbf{e}_z \quad \Rightarrow \quad \mathbf{e}_t = \frac{\mathbf{e}_r + 2ar_0\mathbf{e}_z}{\sqrt{1 + 4a^2r_0^2}}\,,$$

where we have normalized the vector to unit length.

If we now find the projection of $\mathbf{v}$ onto $\mathbf{e}_t$, we can subtract this projection twice, which gives the vector $\mathbf{v}'$ describing the reflected ray (in the opposite direction):

$$\mathbf{v}\cdot\mathbf{e}_t = \frac{r_0 + 2ar_0(ar_0^2 - h)}{\sqrt{1 + 4a^2r_0^2}}\,,$$

thus

$$\begin{aligned}\mathbf{v}' &= \mathbf{v} - 2\frac{r_0 + 2ar_0(ar_0^2 - h)}{\sqrt{1 + 4a^2r_0^2}}\mathbf{e}_t = r_0\mathbf{e}_r + (ar_0^2 - h)\mathbf{e}_z - 2\frac{r_0 + 2ar_0(ar_0^2 - h)}{\sqrt{1 + 4a^2r_0^2}}\frac{\mathbf{e}_r + 2ar_0\mathbf{e}_z}{\sqrt{1 + 4a^2r_0^2}} = \\ &= \left(r_0 - 2\frac{r_0 + 2ar_0(ar_0^2 - h)}{1 + 4a^2r_0^2}\right)\mathbf{e}_r + \left((ar_0^2 - h) - 4ar_0\frac{r_0 + 2ar_0(ar_0^2 - h)}{1 + 4a^2r_0^2}\right)\mathbf{e}_z = \\ &= \frac{r_0(4ah - 1)}{1 + 4a^2r_0^2}\mathbf{e}_r + \frac{-ar_0^2(3 + 4a^2r_0^2) + h(4a^2r_0^2 - 1)}{1 + 4a^2r_0^2}\mathbf{e}_z\,.\end{aligned}$$

The equation of the reflected ray has the form $r = A + \lambda \mathbf{v}'$, therefore

$$r = r_0 + \lambda \frac{r_0(4ah - 1)}{1 + 4a^2 r_0^2}\,, \qquad z = ar_0^2 + \lambda \frac{-ar_0^2(3 + 4a^2 r_0^2) + h(4a^2 r_0^2 - 1)}{1 + 4a^2 r_0^2}\,. \tag{27}$$

This line must intersect the line $z = ar_0^2 + d$, giving the condition for λ

$$ar_0^2 + d = ar_0^2 + \lambda \frac{-ar_0^2(3 + 4a^2 r_0^2) + h(4a^2 r_0^2 - 1)}{1 + 4a^2 r_0^2} \qquad \Rightarrow$$

$$\Rightarrow \qquad \lambda = \frac{d(1 + 4a^2 r_0^2)}{-ar_0^2(3 + 4a^2 r_0^2) + h(4a^2 r_0^2 - 1)}\,.$$

Substituting into the relation for r in (27), we obtain the desired radius r of the circle on the screen

$$r = r_0 + \frac{dr_0(4ah - 1)}{-ar_0^2(3 + 4a^2 r_0^2) + h(4a^2 r_0^2 - 1)}\,. \tag{28}$$

From the form of the vector $\mathbf{v}'$ we can also easily determine the value of h for which we are at the focus of the parabola. In that case the reflected ray must be vertical and the r-component must vanish:

$$\frac{r_0(4ah - 1)}{1 + 4a^2 r_0^2} = 0 \quad \Rightarrow \quad 4ah - 1 = 0 \quad \Rightarrow \quad h_0 = \frac{1}{4a}\,.$$

In that case clearly $r = r_0$, which matches (28). However, we are interested in the dependence on the shift $\mathrm{d}h = h - h_0$. Substitute h into (28):

$$\begin{aligned} r &= r_0 + \frac{4adr_0\,\mathrm{d}h}{-ar_0^2(3 + 4a^2 r_0^2) + \left(\frac{1}{4a} + \mathrm{d}h\right)(4a^2 r_0^2 - 1)} = \\ &= r_0 + \frac{4adr_0\,\mathrm{d}h}{-ar_0^2(3 + 4a^2 r_0^2) + \frac{1}{4a}(4a^2 r_0^2 - 1)} + \mathcal{O}(\mathrm{d}h^2) \end{aligned}$$

and we obtain

$$\mathrm{d}r = -\frac{16a^2 r_0 d}{1 + 8a^2 r_0^2(1 + 2a^2 r_0^2)}\,\mathrm{d}h = -\frac{16v^2 r_0 d}{r_0^4 + 8v^2(r_0^2 + 2v^2)}\,\mathrm{d}h = 1.3\,\mathrm{cm}\,.$$

Problem R.4 ... spotlight at full power 7 points

A headlight in a car consists of a parabolic mirror formed by rotating the parabola $z = v\,(r/r_0)^2$, $r < r_0 = 7.0\,\mathrm{cm}$, where $v = 5.0\,\mathrm{cm}$ about its axis. A negligibly small, incoherent light source with radiant intensity $I_\Omega = 0.4\,\mathrm{W{\cdot}sr^{-1}}$ in the visible spectrum is located at its focus. Assume that at a distance $d = 1.0\,\mathrm{m}$ from the upper edge of the reflector (the plane $z = v$), a screen is placed perpendicular to the paraboloid axis. A bright circular region is visible on the screen, illuminated both by rays reflected from the paraboloid and by rays arriving directly from the source. Determine the magnitude of the jump in intensity at the boundary of this circle.

Kuba was thinking about physics while driving.

Both the rays coming directly from the source and the rays reflected by the paraboloid will fall onto the screen. Since the light is incoherent, the corresponding intensities on the screen will simply add. The jump observed at the edge of the circle is therefore caused only by the intensity of the reflected radiation.

Let us consider spherical coordinates (r, θ, φ) with the origin at the focal point, for which we have derived that it lies at the point $F = [0, 1/(4a)]$, where $a = v/r_0^2$. On the screen, we consider polar coordinates (R, φ), where φ is evidently the same in both coordinate systems.

We will need to know the radius R on the screen at which a ray emitted at an angle θ, measured from the axis of the paraboloid, is imaged. First, let us find the point on the parabola where such a ray is reflected. The direction vector of the incoming ray is clearly

$$\mathbf{v} = \cos\left(\frac{\pi}{2} - \theta\right)\mathbf{e}_x + \sin\left(\frac{\pi}{2} - \theta\right)\mathbf{e}_y = \sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y$$

and therefore the parametric equation of the ray is

$$\begin{aligned} r &= \lambda\sin\theta \\ z &= \frac{1}{4a} + \lambda\cos\theta\,. \end{aligned}$$

This line must intersect the parabola $z = v\,(r/r_0)^2 = a\,r^2$, i.e.,

$$\begin{aligned} ar^2 &= \frac{1}{4a} + \lambda\cos\theta \\ a\lambda^2\sin^2\theta &= \frac{1}{4a} + \lambda\cos\theta \\ \Rightarrow\quad \lambda &= \frac{\cos\theta \pm 1}{2a\ \sin^2\theta}\,, \end{aligned}$$

and since we require $\lambda \geq 0$, we consider the positive value solution only. The ray is therefore reflected at the radius

$$R = \frac{\cos\theta + 1}{2a\sin^2\theta}\sin\theta = \frac{1 + \cos\theta}{2a\sin\theta}\,, \tag{29}$$

and since it reflects as a vertical ray, it hits the screen at this same radius.

We now want to determine the size of the surface element $\mathrm{d}S = R\,\mathrm{d}R\,\mathrm{d}\varphi$ on the screen that is illuminated when rays are emitted from the bulb into the solid angle $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. From equation (29) we have

$$\mathrm{d}R = \mathrm{d}\left(\frac{1 + \cos\theta}{2a\sin\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1 + \cos\theta}{2a\sin\theta}\right)\mathrm{d}\theta = -\frac{1 + \cos\theta}{2a\sin^2\theta}\,\mathrm{d}\theta\,,$$

and therefore the illuminated surface element is

$$\mathrm{d}S = R\,\mathrm{d}R\,\mathrm{d}\varphi = \frac{1 + \cos\theta}{2a\sin^2\theta}R\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{1 + \cos\theta}{2a\sin^3\theta}R\,\mathrm{d}\Omega\,,$$

where we take the absolute value, because rays cross after reflection as θ increases. We are interested only in the absolute surface element.

From the problem statement, we know the radiant intensity of the bulb $I_\Omega = \mathrm{d}P/\,\mathrm{d}\Omega$. Thus, we can compute the power incident onto the area $\mathrm{d}S$ as

$$\mathrm{d}P = I_\Omega\,\mathrm{d}\Omega = I_\Omega\frac{2a\sin^3\theta}{1 + \cos\theta}\frac{\mathrm{d}S}{R} \quad\Rightarrow\quad I = \frac{\mathrm{d}P}{\mathrm{d}S} = I_\Omega\frac{4a^2\sin^4\theta}{(1 + \cos\theta)^2}\,, \tag{30}$$

where we have obtained the resulting intensity I.

It remains to find the angle θ corresponding to the radius $R = r_0$. To do that, we must invert the relation (29). Let us rewrite the trigonometric functions using the angle $\theta/2$

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$$
$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)\,.$$

Substituting yields

$$R = \frac{1 + \cos^2(\theta/2) - \sin^2(\theta/2)}{4a\sin(\theta/2)\cos(\theta/2)} = \frac{2\cos^2(\theta/2)}{4a\sin(\theta/2)\cos(\theta/2)} = \frac{1}{2a}\cot(\theta/2)\,,$$

and the inverse on the interval $(0, 2\pi)$ is

$$\theta = 2\,\mathrm{arccot}(2a\ R)\,.$$

Similarly, we can rewrite the ratio in expression (30)

$$\frac{4a^2\sin^4\theta}{(1+\cos\theta)^2} = \frac{64a^2\sin^4(\theta/2)\cos^4(\theta/2)}{(1+\cos^2(\theta/2)-\sin^2(\theta/2))^2} = \frac{64a^2\sin^4(\theta/2)\cos^4(\theta/2)}{4\cos^4(\theta/2)} = 16a^2\sin^4(\theta/2).$$

Thus, the final expression becomes

$$I = 16a^2 I_\Omega \sin^4\mathrm{arccot}(2aR)\,.$$

Using the relations

$$\sin\mathrm{arccot}\,x = \frac{1}{\sqrt{1+x^2}}\,,$$
$$\cos\mathrm{arccot}\,x = \frac{x}{\sqrt{1+x^2}}\,,$$

we can finally write the result at the point $R = r_0$ as

$$I = 16a^2 I_\Omega \frac{1}{(1+4a^2r_0^2)^2} = \frac{16v^2 I_\Omega}{(r_0^2+4v^2)^2} \doteq 7.2\,\mathrm{mW{\cdot}cm^{-2}}\,.$$

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