Solutions of Physics Brawl Online 2024

Problem 1 ... fair sprint — 3 points

At this year's Olympic Games in Paris, the men's 100 m *sprint was won by American Noah Lyles in* 9.784 s*, Jamaica's Kishane Thompson finished second by five thousandths of a second. Lyles ran on the track number 7, Thompson on the track number 4. Sprint is started by the report of the starter's gun firing. Every runner has a speaker right behind their starting blocks to ensure they all hear the sound at the same time. However, if they started at the sound of a real gunshot fired from a pistol located near lane 1 at the start line, by how much time would Thompson have won? The width of a lane is* 1.22 m*.*

Jarda thought that Forrest Gump had won.

The speed of sound is $343\,\mathrm{m\cdot s^{-1}}$ and the distance of both sprinters' left ears is $3\cdot 1.22\,\mathrm{m} = 3.66\,\mathrm{m}$. We get the time difference

$$\Delta t = \frac{3.66\,\mathrm{m}}{343\,\mathrm{m\cdot s^{-1}}} \doteq 0.0107\,\mathrm{s}\,.$$

Thus, the Jamaican would hear the gunshot about one hundredth of a second earlier than Lyles, therefore starting earlier and crossing the finish line first. He would win by

$$\Delta t_{\mathrm{win}} = 0.01067\,\mathrm{s} - 0.005\mathrm{s} \doteq 0.006\,\mathrm{s}\,,$$

so the race would end up similarly, only with the opposite result.

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Problem 2 ... rail problem — 3 points

For a more comfortable train ride, the continuous welded rail *technology, which consists of welding the rails and creating a uniform surface for the train to pass, is used. However, such a track must withstand thermal expansion in summer and winter. What temperature difference can a steel rail withstand if its linear thermal expansion coefficient is* $1.63\cdot 10^{-5}\,\mathrm{K^{-1}}$ *and the permitted stress of the rail is* 600 MPa*? Assume that the tensile Young's modulus for steel is* 195 GPa*.*

David took a train to the dorms.

To determine how big of a temperature difference the rail can withstand, we will use the Hooke's law. This law states that the normal stress σ is directly proportional to the relative elongation ε through the Young's modulus of elasticity E

$$\sigma = E\varepsilon\,.$$

Relative elongation is defined as the ratio of the variation in length, Δl, to the original length, l_0

$$\varepsilon = \frac{\Delta l}{l_0}\,.$$

Next, we need to examine how the rail length would change with a temperature change of Δt. In reality, the length of the rail remains constant, resulting in stress within the rail, which adheres to the same relations. The elongation can be expressed using the formula

$$\Delta l = l_0\alpha\Delta t\,,$$

where α is the coefficient of thermal expansion. From here, we can derive that the temperature difference Δt_m at which the rail exceeds the allowable stress σ_m is given as

$$\Delta t_\mathrm{m} = \frac{\sigma_\mathrm{m}}{E\alpha}\,.$$

After substitution, we find that $\Delta t_\mathrm{m} \doteq 189\,\mathrm{K}$. To justify the equations used in the calculation, we can also picture the scenario as if first, the rail extends, but then we compress it back to its original size. In practice, the rails will begin to bend even at lower temperatures, particularly due to the shape of the rails and the weight of the trains acting upon them.

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Problem 3 ... in the shadows

3 points

Consider a street lamp of height $H = 3.2\,\mathrm{m}$ as a point source of light. A person of height $h = 1.8\,\mathrm{m}$ is walking in a straight line directly away from the lamp at a constant speed of $v = 1.5\,\mathrm{m\cdot s^{-1}}$. Calculate the acceleration of the top of their shadow.

Marek lives in the shadows.

Let us denote H the height of the lamp and h the height of the man. The distance of the man from the lamp is s, and x is the length of his shadow on the ground. Next, let us denote the initial distance of the man from the lamp as s_0. Then the similarity of the triangles implies

$$\frac{s+x}{H} = \frac{x}{h} \quad \Rightarrow \quad x = \frac{sh}{H-h}\,.$$

The distance of the person from the lamp varies in time as $s = vt + s_0$, so we obtain

$$x = \frac{h(vt+s_0)}{H-h} = \frac{hv}{H-h}t + \frac{hs_0}{H-h}\,.$$

We see that the tip of the shadow moves in a uniform linear motion whose acceleration is therefore zero.

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Problem 4 ... run as fast as you can

3 points

A pilot of a Mitsubishi A6M Zero fighter aircraft with a power of $P = 940\,\mathrm{hp}$ was flying horizontally over an atoll belonging to the United States of America at speed $v = 180\,\mathrm{kn}$. When the pilot spotted the runway with aircrafts attempting to take off, he pulled the triggers of both $7.7\,\mathrm{mm}$ fixed-mounted Type 97 aircraft machine guns with a cadence of $c = 900\,\mathrm{min^{-1}}$ and a muzzle velocity of $u = 745\,\mathrm{m\cdot s^{-1}}$. Each of the machine guns had one full belt of ammunition with 500 rounds. If the pilot flew exactly over the runway that was $l = 7\,800\,\mathrm{ft}$ long, how many rounds did he hit it with?

Dodo watched war movie.

Since the aircraft flies horizontally, at a constant height above the runway, the trajectories of the projectiles will intersect the runway at the same intervals of distance at which they left the barrel. The time that Zero takes to fly across the runway is

$$t = \frac{s}{v} = \frac{7\,800\,\mathrm{ft}}{180\,\mathrm{kn}} = \frac{2\,377\,\mathrm{m}}{92.6\,\mathrm{m\cdot s^{-1}}} = 25.67\,\mathrm{s}\,.$$

The machine gun can fire continuously for a maximum of $500/900\,\mathrm{min} = 33.3\,\mathrm{s}$. It therefore has enough rounds to cover the entire runway. So from one machine gun a total of

$$N = 25.67\,\mathrm{s} \cdot 900\,\mathrm{min}^{-1} = 385\,\mathrm{rounds}\,,$$

hit the runway. However, there are two machine guns, so the answer is $N \doteq 770$.

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Problem 5 ... water resource engineering 3 points

Water should be conserved, which is why we collect rainwater into a barrel. Water flows into the barrel via a vertical gutter, which collects water from the roof. When the barrel gets full, water is wastefully spilled into the surrounding garden. We would like to redirect the water flow outside the barrel when it gets full, instead of overfilling the barrel. Verča's dad is an engineer (just like she will be one day) and invented a clever machine to solve this problem. On the surface floats a plastic cylinder with a base of area $S = 100\,\mathrm{cm}^2$ and density $\rho = 550\,\mathrm{kg{\cdot}m^{-3}}$, connected via rigid stick to the gutter end. When the surface rises, it pushes the cylinder up, eventually closing the gutter end.

What is the minimal height of the cylinder in order for the buoyancy to be strong enough to close the gutter end while being heavy enough to let the end open as the surface lowers? Assume that the force $F = 15\,\mathrm{N}$ is required in order to open or close the gutter end (for closing up, for opening down), and otherwise the system cylinder-gutter end does not move.

Verča watched her daddy at work.

We will write out the conditions for both cases (opening and closing of the outlet).

To open the gutter, we need

$$F < Sh\rho g\,,$$

where h is the height of the cylinder.

To close it, we must satisfy the condition

$$F < Shg(\rho_\mathrm{v} - \rho)\,,$$

where $\rho_\mathrm{v} = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water and $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ is the gravitational acceleration. We can see that in both cases, the acting force is directly proportional to the height of the cylinder. Therefore, it suffices to determine in which case h is greater; this will be the necessary minimum height. After expressing from both conditions, we obtain

$$h_1 > 27.8\,\mathrm{cm}\,,$$
$$h_2 > 34.1\,\mathrm{cm}\,.$$

From this, we see that the minimum height of the cylinder with the given parameters is 34.1 cm.

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Problem 6 ... cooking up acid

4 points

What is the minimum mass of the sulfuric acid H_2SO_4 that we need to mix with $200\,\mathrm{ml}$ *of pure water to bring the mixture to the boiling point, which is* $98\,{}^\circ\mathrm{C}$ *in the laboratory? The initial temperature of both liquids is* $25\,{}^\circ\mathrm{C}$*. When mixed* $70.5\,\mathrm{kJ}$ *of heat is released per mole of sulfuric acid added.*

Assume a specific heat capacity of the resulting solution of $0.75\,\mathrm{cal{\cdot}g^{-1}{\cdot}{}^\circ C^{-1}}$ *independent of external conditions, and with molar masses of hydrogen* $1.01\,\mathrm{g{\cdot}mol^{-1}}$*, oxygen* $16.00\,\mathrm{g{\cdot}mol^{-1}}$ *and sulfur* $32.07\,\mathrm{g{\cdot}mol^{-1}}$*. Assume that the reaction happens quickly and under standard conditions. Neglect the heat capacity of the container and any heat losses. Next consider a simplified case, where the amount of heat released per mole of added acid remains constant until the mole ratio of acid to water exceeds* 0.1*. At that point, the amount of heat produced by adding more acid will decrease significantly, and therefore assume that in this case, the mixture will not reach the boiling point. If this occurs, give the result as* $0\,\mathrm{g}$*.*

Karel was wondering, why is the mixing so dangerous.

The total heat Q released is determined as

$$Q = n_{\mathrm{H_2SO_4}} Q_\mathrm{m} = \frac{m_{\mathrm{H_2SO_4}}}{M_{\mathrm{H_2SO_4}}} Q_\mathrm{m}\,,$$

where $n_{\mathrm{H_2SO_4}}$ is the amount of substance of $\mathrm{H_2SO_4}$, which is expressed as the ratio of its mass $m_{\mathrm{H_2SO_4}}$ to its molar mass $M_{\mathrm{H_2SO_4}}$, and Q_m is the molar heat released during the dilution of $\mathrm{H_2SO_4}$. The molar mass of the molecule is the sum of the molar masses of all the atoms in it, so the following holds

$$M_{\mathrm{H_2SO_4}} = 2M_\mathrm{H} + M_\mathrm{S} + 4M_\mathrm{O}$$

where M_H, M_S, and M_O are the molar masses of hydrogen, sulfur, and oxygen, respectively, as provided in the problem statement.

According to the assumptions and the law of conservation of energy, all the heat Q is utilized to heat up the solution, and then, by the law of conservation of mass, we have

$$Q = \left(m_{\mathrm{H_2SO_4}} + m_{\mathrm{H_2O}}\right) c\,(t_\mathrm{v} - t_0) = \left(m_{\mathrm{H_2SO_4}} + \rho_{\mathrm{H_2O}} V_{\mathrm{H_2O}}\right) c\,(t_\mathrm{v} - t_0)$$

where $m_{\mathrm{H_2O}}$ is the mass of water expressed through its volume $V_{\mathrm{H_2O}}$ and density $\rho_{\mathrm{H_2O}} = 0.998\,\mathrm{g{\cdot}cm^{-3}}$, $c = 3.138\,\mathrm{J{\cdot}g^{-1}{\cdot}{}^\circ C^{-1}}$ is the specific heat capacity of the solution, t_v is the boiling point of the mixture and t_0 is its initial temperature. From the heat equation, we need to express the sought mass of $\mathrm{H_2SO_4}$, and then substitute the values from the problem statement and the list of constants. We obtain

$$m_{\mathrm{H_2SO_4}} = \frac{\rho_{\mathrm{H_2O}} V_{\mathrm{H_2O}} c\,(t_\mathrm{v} - t_0)}{\frac{Q_\mathrm{m}}{2M_\mathrm{H} + M_\mathrm{S} + 4M_\mathrm{O}} - c\,(t_\mathrm{v} - t_0)} \doteq 93\,\mathrm{g}\,.$$

Finally, we verify the condition given in the problem statement. For the ratio p of the amount of substance of acid $n_{\mathrm{H_2SO_4}}$ to the amount of substance of water $n_{\mathrm{H_2O}}$ holds the following

$$p = \frac{n_{\mathrm{H_2SO_4}}}{n_{\mathrm{H_2O}}} = \frac{m_{\mathrm{H_2SO_4}} M_{\mathrm{H_2O}}}{m_{\mathrm{H_2O}} M_{\mathrm{H_2SO_4}}} = \frac{m_{\mathrm{H_2SO_4}}\,(2M_\mathrm{H} + M_\mathrm{O})}{\rho_{\mathrm{H_2O}} V_{\mathrm{H_2O}}\,(2M_\mathrm{H} + M_\mathrm{S} + 4M_\mathrm{O})} \doteq 0.09 < 0.1\,,$$

and therefore the above-calculated solution is valid.

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Problem 7 ... planespotting 3 points

Martin is looking out of the window and sees a flying plane. He wonders, how fast the plane could be flying. In his mind, Martin counts that the plane was in his field of view between two houses for 10 seconds (the aircraft is in the field of view if Martin can see at least half of the aircraft). According to the rule "1 thumb in the distance of an outstretched hand corresponds to 2.5 *angular degrees" he calculates that the plane has flown* 7.5 *of his "thumbs" during that time. Martin doesn't know how far the plane was flying, but he makes an estimation, that in his field of view, it occupied about half of his thumb during the whole observation and that the plane could be a Boeing 737-800. Calculate, how fast the plane was flying, based on Martin's estimations.*

Martin was looking out of Veronika's window.

Let us first review which quantities we know:

- Aircraft observation time: $t = 10\,\mathrm{s}$
- Angular crossing distance: $7.5\,\text{inches} \cdot 2.5\,^\circ$ per inch $= 18.75\,^\circ$
- Angular size of the aircraft: $0.5\,\text{inches} \cdot 2.5\,^\circ$ per inch $= 1.25\,^\circ$

We do not know how far the plane is to calculate its flight path using the angle it flew. However, we can easily calculate it if we look up the length of the Boeing 737-800, which is $l = 39.5\,\mathrm{m}$. The trajectory s of the aircraft is then from the cross multiplication

$$s = \frac{18.75\,^\circ}{1.25\,^\circ} \cdot 39.5\,\mathrm{m} = 592.5\,\mathrm{m}\,.$$

We then calculate the velocity trivially as

$$v = \frac{s}{t} = \frac{592.5\,\mathrm{m}}{10\,\mathrm{s}} = 59.25\,\mathrm{m{\cdot}s^{-1}} = 213.3\,\mathrm{km{\cdot}h^{-1}}\,.$$

Hence, the plane would be flying at about $213\,\mathrm{km{\cdot}h^{-1}}$. That is much less than the top speed of a Boeing 737-800, which is about $950\,\mathrm{km{\cdot}h^{-1}}$. Martin's problem statement omitted the fact that he observed the plane as it landed, but still if it was taken into account, it is an inaccurate result, yet, given the instruments used (thumbs), it is a quite reasonable order of magnitude estimate (planes normally land at speeds of around $250\,\mathrm{km{\cdot}h^{-1}}$ – $300\,\mathrm{km{\cdot}h^{-1}}$).

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Problem 8 ... formula car on the ceiling 4 points

To maximize the frictional force between the wheels of Formula 1 cars and the road surface, the car body is designed so that air resistance pushes the car to the ground. At what speed would the Formula 1 cars have to travel in order to drive heads down on the ceiling? For simplicity, consider that the car is a right-angled triangle of length $5.1\,\mathrm{m}$, *height* $1.0\,\mathrm{m}$, *and width* $1.8\,\mathrm{m}$. *The vehicle's mass, including the driver, is* $800\,\mathrm{kg}$. *Let us further assume that the air particles are initially stationary and elastically bounce off the vehicle.*

Jarda wanted to skip a traffic jam in a tunnel.

We will examine the situation from the inertial frame of reference attached to the Formula 1 car. In this frame, the air molecules move with a velocity v relative to the formula. Since the collision is elastic and the mass of the formula is much greater than that of the air molecules, we can assume that for the colliding molecules, the angle of reflection is equal to the angle of incidence; let us denote this angle as α.

Furthermore, since the molecules move along a straight line parallel to the roof, we can determine this angle from the shape of the vehicle (a right triangle) as

$$\alpha = \arctan \frac{h}{d} \doteq 11.3\,^{\circ} ,$$

where $h = 1.0\,\mathrm{m}$ represents the height of the right triangle and $d = 5.1\,\mathrm{m}$ the length of its base.

After the collision, the molecules will have the same speed but will be deflected from their original direction by an angle of 2α (angle of incidence + angle of reflection). For their momentum in the direction perpendicular to the roof, we have

$$p_y = mv \sin 2\alpha .$$

From the law of conservation of momentum, this momentum is equal to the momentum transferred to the vehicle in this direction during the collision. For the total momentum over time, we need to substitute the mass of all the molecules colliding with the vehicle in time $\mathrm{d}t$ into the relation above.

Let us consider a rectangular prism with dimensions $w = 1.8\,\mathrm{m}$ (the width of the Formula 1), d, and $h\,\mathrm{d}t$, where $h\,\mathrm{d}t$ equals the distance traveled by the formula in time $\mathrm{d}t$.

The mass is then given as

$$\mathrm{d}m = hwd\rho\,\mathrm{d}t ,$$

where ρ is the density of air.

Substituting into the momentum equation, we get

$$\mathrm{d}p_y = h \sin 2\alpha\,\mathrm{d}m ,$$
$$\mathrm{d}p_y = wdv^2\rho \sin 2\alpha\,\mathrm{d}t .$$

Finally, the relation between momentum and the upward force acting on the vehicle is

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} ,$$

from which

$$F_y = sdv^2\rho \sin 2\alpha .$$

This force must balance out the gravitational force Mg acting on the formula

$$Mg = whd^2\rho \sin 2\alpha ,$$

where M is the mass of the formula.

Rearranging gives

$$v = \sqrt{\frac{Mg}{wd\rho \sin 2\alpha}} \doteq 98\,m \cdot s^{-1} .$$

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Problem 9 ... warming up a baking tray with pendulum 4 points

The induced currents in a conductor can be demonstrated by an experiment in which a metal tray is placed under a swinging pendulum consisting of a magnet attached to a string and the oscillations are quickly damped. By how many degrees does such a tray heat up? We use an aluminum tray of width and length $a = 12\,\mathrm{cm}$, *thickness* $d = 0.42\,\mathrm{mm}$ *with specific heat capacity* $c_{\mathrm{Al}} = 896\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. *The pendulum suspension has a length* $l = 84\,\mathrm{cm}$, *a maximum deflection* $\alpha = 33\,^{\circ}$ *and the mass of the weight is* $m = 120\,\mathrm{g}$. *Consider that half of the energy of the pendulum is converted into heat, which is received by the metal tray, and the other half is dissipated by other means. The heat is distributed evenly in the tray. Neglect the weight of the string and the dimensions of the magnet. The density of aluminum is* $\rho_{\mathrm{Al}} = 2\,700\,\mathrm{kg\cdot m^{-3}}$.

Karel saw a picture of a demonstrative experiment in a textbook.

The initial energy of the pendulum is equal to its potential energy, which can be evaluated as

$$E_{\mathrm{p}} = mgl\left(1 - \cos\alpha\right) .$$

Half of this energy is used for the heating up of the metal tray according to the formula

$$Q = a^2 d c_{\mathrm{Al}} \rho_{\mathrm{Al}} \Delta T\,,$$

where ΔT is difference between the final and the initial temperature and $\rho_{\mathrm{Al}} = 2\,700\,\mathrm{kg\cdot m^{-3}}$ is the density of aluminum. Rearranging the formula we get

$$\frac{1}{2} mgl\left(1 - \cos\alpha\right) = a^2 d c_{\mathrm{Al}} \rho_{\mathrm{Al}} \Delta T\,,$$

$$\Delta T = \frac{mgl(1-\cos\alpha)}{2a^2 d c_{Al}\rho_{Al}}\,,$$

$$\Delta T = 5.5\cdot 10^{-3}\,\mathrm{K}\,.$$

Therefore, the metal tray has heated up by $\Delta T = 5.5\cdot 10^{-3}\,\mathrm{K}$.

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Problem 10 ... diproton bids farewell 4 points

Let us imagine a particle composed of two protons, i.e., a helium 2 nucleus. Such a particle is highly unstable and rapidly decays into two separate protons. What is the maximum speed that these protons can acquire after a spontaneous decay in the center-of-momentum frame? Consider that the rest mass of a diproton is $2.015\,89\,\mathrm{Da}$ *and the mass of a single proton is* $1.007\,825\,\mathrm{Da}$.

Karel wondered about a problem with helium 2.

The key is to realize that in the center-of-momentum frame, according to the law of conservation of momentum, the protons must move with equal speeds in opposite directions after the decay. We calculate this speed using the law of conservation of energy.

The energy released by the decay is given by the equation

$$E = (m_{\mathrm{d}} - 2m_{\mathrm{p}})c^2\,,$$

where m_d is the mass of a diproton and m_p is the mass of a proton. This energy is converted into kinetic energy

$$E = 2 \cdot \frac{1}{2} m_\mathrm{p} v^2 \,,$$

where v is the velocity of one proton.

For v, we then get

$$v = c\sqrt{\frac{m_\mathrm{d}}{m_\mathrm{p}} - 2}$$

or numerically, $v = 4.63 \cdot 10^6\,\mathrm{m{\cdot}s^{-1}}$. We see that the velocity of the protons is approximately 1.5 percent of the speed of light, which indicates that the non-relativistic approximation provides a fairly accurate result.

Note: Dalton is a unit of mass equal to the atomic mass constant.

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Problem 11 ... accelerated deceleration 4 points

Jarda filmed his route around the city on a camera in his car. At home, he then began to examine a section of the footage where he was traveling at the maximum allowed speed of $50\,\mathrm{km{\cdot}h^{-1}}$*, and* $55\,\mathrm{m}$ *before the intersection, the traffic light flashed red, so he slowed down with constant acceleration to stop at the intersection. How many times faster does he have to play this recording to make it appear as if he was slowing down with a drastic acceleration of* $10g$*?* *Jarda wanted to prove he was a superhero.*

Let us denote $s = 55\,\mathrm{m}$ as a breaking distance, a_r as the acceleration used to stop the car, and the speed as $v_\mathrm{r} = 50\,\mathrm{km{\cdot}h^{-1}} \doteq 13.9\,\mathrm{m{\cdot}s^{-1}}$. According to the law of conservation of energy

$$Fs = ma_\mathrm{r}s = \frac{1}{2}mv_\mathrm{r}^2 \quad \Rightarrow \quad a_\mathrm{r} = \frac{v_\mathrm{r}^2}{2s} \doteq 1.75\,\mathrm{m{\cdot}s^{-2}} \,,$$

where we compared the car's initial kinetic energy with the work exerted by the breaking force F to stop the vehicle.

Now assume that the footage is replayed x-times faster. The car is therefore moving with the speed $v_\mathrm{a} = xv_\mathrm{r}$. The distance s is the same as on the original footage. Using the formula above, we can find the condition for the acceleration

$$10g = a_\mathrm{a} = \frac{v_\mathrm{a}^2}{2s} = x^2\frac{v_\mathrm{r}^2}{2s} \quad \Rightarrow \quad x = \sqrt{10g\frac{2s}{v_\mathrm{r}^2}} = 7.5 \,.$$

The footage must be sped up by at least seven times to achieve the desired acceleration.

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Problem 12 ... a cracked bottle

4 points

A bottle of sparkling water slipped out of Eliška's hand and fell to the ground. When it hit the floor, a hole was made in the bottle, from which the drink began to spurt. Consider a model situation where water sprays out at an angle $\alpha = 70\,^\circ$ (measured from the ground) to a height of $h = 1.20\,\mathrm{m}$. What was the initial difference between the pressure in the bottle and the ambient pressure? *Eliška dropped a bottle of Kofola.*

From the law of conservation of energy, we determine the initial vertical velocity of the water

$$\frac{1}{2}mv_y^2 = mgh\,,$$

$$v_y = \sqrt{2gh}\,.$$

The total velocity is then calculated as

$$v = \frac{v_y}{\sin{(\alpha)}}\,.$$

Now, to relate the pressure to this velocity, we use Bernoulli's equation, which states:

$$p + \frac{1}{2}\rho v^2 = \mathrm{const}\,.$$

For the region between the inside of the bottle, where the water has zero velocity, and the water outside, where the pressure is equal to the atmospheric pressure, we have

$$p + p_\mathrm{a} = \frac{1}{2}\rho v^2 + p_\mathrm{a}\,.$$

We substitute for the velocity

$$p = \frac{1}{2}\rho\frac{2gh}{sin^2(\alpha)}\,,$$

from which

$$p \doteq 1.3\cdot 10^4\,\mathrm{Pa}\,.$$

The pressure in the bottle immediately after the impact was approximately $1.3 \cdot 10^4\,\mathrm{Pa}$ higher than the atmospheric pressure.

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Problem 13 ... big guitar bang

4 points

As Paťo walked through the door, he slammed his guitar against the doorjamb. Later on, when he checked the damage, he noticed that one of the strings was tuned exactly a semitone higher than the original frequency. Since the string was already due for replacement before that, Pato figured he could kill two birds with one stone.

He decided to tightly and firmly wind a new layer of wire onto the string in the shape of a helix. How thick of a wire, made from the same material, does he need to buy to retune the string to its original frequency? The problematic string had a thickness of $1.1\,\mathrm{mm}$ at the start. Assume equal temperament tuning and that the tension in the string remains unchanged during *this manipulation. Consider the string both in its original state and after the wire has been wound, as a homogeneous circular right cylinder.* *Paťo didn't want to change the strings.*

Let us start with a bit of music mathematics. In equal temperament tuning, the frequencies of semitones form a geometric sequence. Semitones group into octaves after *twelve* steps, and a semitone one octave higher has *double* the frequency of the corresponding lower semitone. Translating this into mathematics, we can derive the common ratio q of the sequence as

$$2 = \frac{f_{n+12}}{f_n} = \frac{f_n \cdot q^{12}}{f_n} = q^{12} \quad \Rightarrow \quad q = \sqrt[12]{2}$$

for any two semitones f_n and f_{n+12} that are an octave apart.

Let f_d denote the string's frequency before detuning (and after adding the wire) and f_r the frequency after detuning. From the previous relation, we have

$$f_\mathrm{r} = q f_\mathrm{d} = \sqrt[12]{2} f_\mathrm{d}\,.$$

The vibrations of the string are governed by Mersenne's laws, which state that a string of length l, with linear mass density μ, under tension F, vibrates at the frequency

$$f = \frac{1}{2l}\sqrt{\frac{F}{\mu}}\,.$$

Before substituting this into the previous equation, we note that wrapping a wire around the string does not change its length l or the tension F (per the problem statement). This gives

$$\frac{1}{2l}\sqrt{\frac{F}{\mu_\mathrm{r}}} = \frac{\sqrt[12]{2}}{2l}\sqrt{\frac{F}{\mu_\mathrm{d}}} \quad \Rightarrow \quad \mu_\mathrm{d} = \sqrt[6]{2}\mu_\mathrm{r}$$

for the string's linear mass densities μ_r (before wrapping) and μ_d (after wrapping).

Assuming the string is a homogeneous cylinder, its linear mass density can be expressed using its volume density and cross-sectional area. Since the material (and thus the volume density ρ) is constant, we find

$$\rho S_\mathrm{d} = \mu_\mathrm{d} = \sqrt[6]{2}\mu_\mathrm{r} = \sqrt[6]{2}\rho S_\mathrm{r} \quad \Rightarrow \quad S_\mathrm{d} = \sqrt[6]{2} S_\mathrm{r}$$

where ρ is the density of the material, and S_r and S_d are the cross-sectional areas of the string before and after wrapping, respectively.

Assuming circular cross-sections, we calculate the diameters. The diameter of the detuned string is given as $d_\mathrm{r} = 1.1\,\mathrm{mm}$, and wrapping the wire increases this by two diameters of the wire d_d

$$\pi\frac{(2d_\mathrm{d} + d_\mathrm{r})^2}{4} = S_\mathrm{d} = \sqrt[6]{2} S_\mathrm{r} = \sqrt[6]{2}\pi\frac{d_\mathrm{r}^2}{4}\,,$$

from which, solving for d_d, we obtain

$$d_\mathrm{d} = \left(\sqrt[12]{2} - 1\right)\frac{d_\mathrm{r}}{2} \doteq 3.3 \cdot 10^{-2}\,\mathrm{mm}\,.$$

Finding such a thin wire might take an eternity, so despite the laziness, replacing and tuning the string seems to be a simpler option.

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Problem 14 ... road on road 4 points

Torrential rains carried away cube-shaped paving stones with an edge of $10\,\mathrm{cm}$ *and a mass of* $2\,\mathrm{kg}$ *from the road. What must have been the minimum depth of the water flow needed to overcome the coefficient of friction* 0.5 *and thus move the cobblestones lying separately on the subgrade? The water was flowing at a speed of* $1.3\,\mathrm{m{\cdot}s^{-1}}$*. Assume drag coefficient of* 1.05*.*

There were floods in Czechia this September.

The frictional force is proportional to the normal force, which in this case is

$$N = mg - a^2 h\rho g\,,$$

where we have subtracted the buoyant force from the weight, h is the height of the water we are looking for and ρ is its density. Even though the cube is on a flat surface, we cannot assume that the water cannot get under the cube, as this surface (e.g. sand) might be porous and not so flat.

The water therefore exerts its hydrostatic pressure on the cube from below.

The resistive force of the water which must overcome the frictional force, is determined as

$$F = \frac{1}{2} C a h \rho v^2\,,$$

where $C = 1.05$ is the drag coefficient and the product ah is the area on which the water acts. From the equality of forces, we obtain

$$\frac{1}{2} C a\ h\rho v^2 = fmg - fa^2 h\rho g \quad \Rightarrow \quad h = \frac{fmg}{a\rho\left(\frac{1}{2}Cv^2 + fa\ g\right)} \doteq 7.1\,\mathrm{cm}\,.$$

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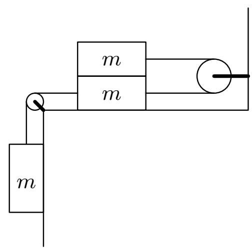
Problem 15 ... pulleys with blocks and friction 5 points

Two blocks lie on top of each other and are connected via a pulley so that the rope runs horizontally in both directions. The bottom block is connected by a rope via a pulley to another block which is hanging on this rope. There is a coefficient of friction $f = 0.10$ *between the blocks and between the blocks and the ground. All three blocks weigh* $m = 500\,\mathrm{g}$*. What will be the magnitude of the acceleration of the hanging block? All the pulleys and ropes have no mass and the system is initially at rest.*

Lego thought... this is definitely a physics problem.



At the beginning, it is necessary to find out which frictional forces are in the system. The upper block exerts a force mg on the lower block, so the frictional force braking this block will be μmg. The lower block will thus be decelerated by the reaction to this force. At the same time, the bottom block will be pressed against the pad by a force of $2\mu mg$, and hence will be additionally decelerated by a frictional force between itself and the pad of magnitude $2\mu mg$, so in total $3\mu mg$.

Since both ropes and pulleys are assumed to be weightless, the tension in the rope will not vary along its length. Let us therefore denote the magnitude of the tension in the rope on which the block hangs as T_1 and the magnitude of the tension in the other rope as T_2.

The magnitudes of the forces acting on the blocks will be as follows: a weight acts on the first block and it is pulled upwards by the rope

$$F_1 = mg - T_1 \,,$$

where the positive sign is chosen for the directions in which each block will move. The second block is accelerated by the first rope and decelerated by the second rope and friction

$$F_2 = T_1 - T_2 - 3\mu mg \,.$$

The last block is accelerated by the second rope and braked only by the frictional force.

$$F_3 = T_2 - \mu mg \,.$$

Since all blocks are "on the same rope", their accelerations will be the same. In other words, when the hanging block moves down by a distance x, the two remaining blocks will also move upwards by the distance x. Thus, the magnitudes of their accelerations must also be equal. Consequently, we get the equality

$$ma = F_1 = F_2 = F_3 \,,$$

which is a system of equations with three unknowns: T_1, T_2, a. We first express T_2

$$\begin{aligned} F_1 &= F_3 \,, \\ mg - T_1 + \mu mg &= T_2 \,, \end{aligned}$$

from which we then express T_1

$$\begin{aligned} F_1 &= F_2 \,, \\ mg - T_1 &= T_1 - T_2 - 3\mu mg \,, \\ mg - T_1 &= T_1 - mg + T_1 - \mu mg - 3\mu mg \,, \\ 2mg + 4\mu mg &= 3T_1 \,, \end{aligned}$$

from which we finally obtain the acceleration as

$$\begin{aligned} ma &= F_1 \,, \\ ma &= mg - \frac{2}{3}mg - \frac{4}{3}\mu mg \,, \\ a &= \frac{1}{3}g - \frac{4}{3}\mu g = 2.0\,\mathrm{m{\cdot}s^{-2}} \,. \end{aligned}$$

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Problem 16 ... increasing the degree of polarization 4 points

The degree of polarization of light P is defined using the intensities of two perpendicular polarizations, I_1 and I_2, as

$$P = \frac{|I_1 - I_2|}{I_1 + I_2}\,.$$

Imagine an environment in which the intensity of the vertically polarized light decreases to 99.0 % after each traveled distance L. Meanwhile, the intensity of the horizontally polarized light decreases faster, to 97.5 % after each distance L. After what distance will the light have a degree of polarization $P = 0.420$? Provide your answer as a multiple of L to one decimal place. Before entering the environment, the light is unpolarized ($P_0 = 0.000$).

Karel was thinking about the anisotropic environment during the election.

From the problem statement, we know that the vertical intensity I_v and the horizontal intensity I_h decrease with distance x according to the relations

$$I_\mathrm{v} = 0.990^{x/L} I_0\,,\quad I_\mathrm{h} = 0.975^{x/L} I_0\,,$$

where I_0 is the initial intensity of each polarization (not the total initial intensity of the light beam, which is twice this value).

We substitute these expressions into the definition of polarization and solve the equation

$$0.420 = \frac{0.990^{x/L} I_0 - 0.975^{x/L} I_0}{0.990^{x/L} I_0 + 0.975^{x/L} I_0} = \frac{0.990^{x/L} - 0.975^{x/L}}{0.990^{x/L} + 0.975^{x/L}}\,,$$

where we see that the result does not depend on the initial intensity I_0. This equation can be solved numerically using a computational tool like WolframAlpha.[1] The result is equal to $x = 58.6L$, so the value 58.6 has to be submitted into the game system.

If you are more mathematically proficient and you can quickly solve exponential relations, you can proceed as follows, though this method is more prone to errors when rewriting the equations.

$$\begin{aligned}
0.420 \cdot \left(0.990^{x/L} + 0.975^{x/L}\right) &= 0.990^{x/L} - 0.975^{x/L} \\
1.420 \cdot 0.975^{x/L} &= 0.580 \cdot 0.990^{x/L} \\
\frac{1.420}{0.580} &= \frac{0.990^{x/L}}{0.975^{x/L}} = \left(\frac{0.990}{0.975}\right)^{x/L} \\
x &= \frac{\ln\frac{1.420}{0.580}}{\ln\frac{0.990}{0.975}} L \doteq 58.6\,L
\end{aligned}$$

Thus, we arrive at the same result $x = 58.6L$. Finally, we could look at how the polarization levels have decreased. The values became $I_\mathrm{v} \doteq 0.555$ and $I_\mathrm{h}(x) \doteq 0.227$

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[1] https://www.wolframalpha.com/input?i=%280.99%5Ex+-+0.975%5Ex%29%2F%280.99%5Ex%2B+0.975%5Ex%29%3D0.420

Problem 17 ... warmer background 5 points

Cosmic background radiation is one of the pieces of evidence for the Big Bang which we can observe today. It corresponds very well to the thermal radiation of a black body with a temperature of $T_0 = 2.725\,\mathrm{K}$*. Given that the universe is* $13.8 \cdot 10^9$ *years old, how many years ago was the temperature of this radiation* $T = 1.1T_0$*? Assume that the universe expands uniformly (the scale factor is directly proportional to time).*

Karel wondered if the universe is infinite... whether to have a toast...

If the universe expands uniformly, then $a \sim t$ holds for the scale factor. The scale factor indicates how much the distances in the universe at a certain time are larger or smaller, compared to the current distances. Just like all other distances, the wavelength of the cosmic background radiation changes as well. Since the radiation corresponds to the emission of a black body, we can use Wien's law relating its temperature and wavelength

$$\lambda = \frac{b}{T}\,,$$

where b is the Wien's constant. Then for the temperature $T \propto 1/\lambda$. From here, we get

$$\frac{T_0}{T} = \frac{\lambda}{\lambda_0}\,.$$

However, the wavelength ratio is

$$\frac{\lambda}{\lambda_0} = \frac{a}{a_0}\,.$$

Finally, for the ratio of the scale factors, the following holds

$$\frac{a}{a_0} = \frac{t}{t_0}\,.$$

By combining the equations, we obtain

$$\frac{T_0}{T} t_0 = t\,,$$

$$\delta t = t_0 - t = t_0\left(1 - \frac{T_0}{T}\right),$$

$$\delta t = 1.25 \cdot 10^9\,\mathrm{years}\,.$$

Note: A more precise calculation for a universe dominated by dark energy yields a time difference of about $1.3 \cdot 10^9$ years. Thus, we can see that the linear approximation gives relatively correct answers for such a small temperature difference. A linear time dependence of the scale factor is a good solution to the Friedmann equations and corresponds to a negatively curved universe, which does not contain any matter (neither substance, radiation, nor dark energy).

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Problem 18 ... combined 4 points

Consider two parallel infinite wires spaced $l = 10\,\mathrm{cm}$ *apart and non-conductive springs of stiffness* $k = 10\,\mathrm{N{\cdot}mm^{-1}}$ *and a rest length* l *stretched between them. The distance between two consecutive springs is* $100\,\mathrm{km}$*. By how many millimetres does the spring elongate when a current of* $20\,\mathrm{A}$ *starts to flow through one of the wires and* $10\,\mathrm{A}$ *through the other in the opposite direction? Assume that the system has reached a completely stabilized state.*

Káťa wanted to come up with a problem that combined multiple branches of physics.

Let us denote the magnitudes of currents flowing in the wires I_1 and I_2 respectively. The magnitude of force per unit length, by which the wires repel each other, can be expressed as

$$|\mathbf{f}_1| = \frac{\mu_0 I_1 I_2}{2\pi d}\,,$$

where d is a distance of the wires in steady state and μ_0 is permeability of vacuum. Now let us look at one wire. Each spring will exert a force on it given by

$$\mathbf{F} = -k(d-l)\,.$$

If we denote the spacing between the springs as δ, the total force from all springs acting on the wire, per unit length, is

$$\mathbf{f}_2 = -\frac{k\;(d-l)}{\delta}\,.$$

If the system is in steady state, the net force of the force acting on the wire is zero. So we get

$$\mathbf{f}_1 + \mathbf{f}_2 = 0\,,$$

$$\frac{\mu_0 I_1 I_2}{2\pi d} - \frac{k(d-l)}{\delta} = 0\,,$$

leading to a quadratic equation for d in the form of

$$2\pi k\; d^2 - 2\pi l k\; d - \mu_0 \delta I_1 I_2 = 0\,,$$

which has two real solutions

$$d = \frac{l}{2} \pm \frac{1}{2}\sqrt{l^2 + \frac{2\mu_0 \delta I_1 I_2}{\pi k}}\,. \tag{1}$$

From the form (1) we can see that the second solution leads to the negative d, mathematically valid but physically unrealistic solution. So the elongation of the spring will be

$$d - l = -\frac{l}{2} + \frac{1}{2}\sqrt{l^2 + \frac{2\mu_0 \delta I_1 I_2}{\pi k}} = 3.9\,\mathrm{mm}.$$

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Problem 19 . . . laser lunch
5 points

In the third episode of the first season of the show The Big Bang Theory, Leslie Winkle heats her lunch using a laser. In the episode, we learn, that it's a COIL laser ($\lambda = 1\,315\,\mathrm{nm}$) *with a power of* $500\,\mathrm{kW}$ *and the heating process takes* $2.60\,\mathrm{s}$. *The heat capacity of her lunch is* $1.00\,\mathrm{kJ{\cdot}K^{-1}}$ *and it needs to be heated by* $60.0\,\mathrm{^\circ C}$, *to taste good. How many moles of photons emitted by the laser during the heating process do not contribute to the heating of her lunch? Assume that if the photon transfers its energy to the lunch, there will be no energy losses.*

Terka was hungry in the lab.

The energy of a single photon can be expressed using the formula

$$E_\mathrm{f} = \frac{hc}{\lambda}\,,$$

where h is Planck's constant, c is the speed of light, and λ is the wavelength of light. Next, we need to express the total energy required to heat a meal

$$E_\mathrm{o} = C\Delta T\,,$$

where C is the heat capacity of the meal, and ΔT represents the temperature change we need to achieve.

It would also be useful to know how much energy the laser emits during its operating time, namely

$$E_\mathrm{l} = P\tau\,,$$

where P is the power of the laser, and τ is the duration for which it heats the meal.

From these equations, we can determine the energy that is not utilized for the heating of the meal, i.e., the total energy losses

$$E_\mathrm{z} = E_\mathrm{l} - E_\mathrm{o}\,,$$

from where we can calculate the number of photons not utilized for heating as

$$N = \frac{E_\mathrm{z}}{E_\mathrm{f}}\,.$$

By combining all these relationships, we obtain

$$N = \lambda\frac{P\tau - C\Delta T}{hc}\,,$$

and upon substitution, we find that the number of particles N and the amount in moles n are

$$N \doteq 8.21 \cdot 10^{24} \quad \Rightarrow \quad n = \frac{N}{N_A} \doteq 13.6\,\mathrm{mol}\,.$$

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Problem 20 ... garage doors

4 points

A garage door weighs $m = 200\,\mathrm{kg}$. To keep the door motor from lifting the entire weight of the door, the door is connected to a steel torsion spring (see the image 1). The spring has an external diameter of $D = 4.1\,\mathrm{cm}$, and the steel wire which forms the spring has a diameter of $d = 5.0\,\mathrm{mm}$. One end of the spring is fixed. The other end is rotating together with the axis in which the cable is fixed to the bottom of the door. The door swings into the garage to a horizontal position. Assume that the door is bending continuously. When the door is fully in the horizontal position, the spring is relaxed. The height of the door is $H = 2.1\,\mathrm{m}$. Determine the initial limiting number of windings N of the spring, for the door motor to exert utmost the force $F = 150\,\mathrm{N}$. For the given spring, the torque force M depends on the angle of rotation φ as

$$M = k\varphi = \frac{Ed^4}{64ND}\varphi\,,$$

where $E = 200\,\mathrm{GPa}$ is Young's modulus of elasticity for steel.

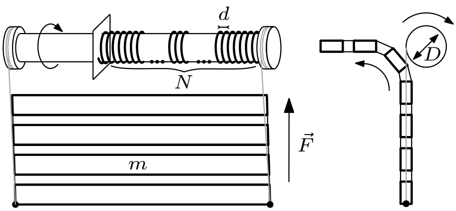

Figure 1: Scheme of the door mechanism.

Jarda couldn't figure out why the garage door wouldn't open when the spring broke.

We have

$$M = k\varphi = \frac{Ed^4}{64ND}\varphi\,.$$

The force acting on the door upwards is $F_\mathrm{p} = 2M/D$. If the door is at the top, the spring is relaxed according to the problem statement. When the door is at the bottom, the angle φ changes according to the diagram by

$$\varphi = 2\pi\frac{H}{\pi D} = \frac{2H}{D}\,,$$

where H is the distance between the ceiling and the bottom of the door.

The force of the spring lifting the door is thus

$$F_\mathrm{p} = \frac{2}{D}\frac{Ed^4}{64ND}\frac{2H}{D} = \frac{Ed^4}{16ND^3}H\,.$$

To lift the door even when it is fully down, the spring force must overcome its weight mg. Our force must therefore be at least

$$F = mg - F_\mathrm{p} = mg - \frac{Ed^4}{16ND^3}H\,.$$

From here, we can express the initial number of coils N as

$$N = \frac{Ed^4}{16D^3\left(mg - F\right)} H \doteq 130\,.$$

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Problem 21 ... who would fear a buoy — 5 points

A seagull weighing $M = 2.0\,\mathrm{kg}$ lands on a cylindrical buoy of mass $m = 5.0\,\mathrm{kg}$ and base area $S = 0.20\,\mathrm{m^2}$. The not very graceful landing makes the buoy rock. Calculate the period of the small oscillations of the buoy if it is oriented vertically, its top base is above the surface of water and it is attached to the bottom of the lake by a taut chain through a spring of stiffness $k = 2.0\,\mathrm{kN{\cdot}m^{-1}}$. The situation takes place on a lake. *Vojta is not afraid of the buoy.*

In the equilibrium state, the weight and the elastic force are balanced by the buoyant force. The relationship is written as

$$(m + M)g + kl_1 = Sl_2\rho g\,,$$

where ρ represents the density of water, and l_1, l_2 are, respectively, the extension of the spring and the height of the submerged part of the cylinder in equilibrium. If the buoy is displaced from this equilibrium position by l (we can assume it is submerged by l), the force F acting on it will have the magnitude

$$\begin{aligned} F &= S(l_2 + l)\rho g - k(l_1 - l) - (m + M)g \\ &= \underbrace{S\ l_2\rho g - k\ l_1 - (m + M)g}_{=0} + Sl\rho g + kl = l(S\rho g + k) \end{aligned}$$

and will be oriented opposite to the direction of the buoy's displacement.

Notice that this relationship mathematically corresponds to Hooke's law – the equation for the force acting on a spring of stiffness $(S\rho g + k)$ displaced by l. From the formula for the period of a spring of a given stiffness and given mass of the weight, we can directly compute

$$T = 2\pi\sqrt{\frac{m + M}{S\rho g + k}} \doteq 0.26\,\mathrm{s}\,.$$

Note that the same relationship could also be derived by solving the equation of motion for this system, but this can be avoided thanks to the mathematical analogy.

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Problem 22 ... rolling glass of ice — 5 points

Consider a tube (cylindrical surface) with very thin walls and a surface density of $\sigma = 5.5\,\mathrm{kg{\cdot}m^{-2}}$. The tube is full of ice with density $\rho = 917\,\mathrm{kg{\cdot}m^{-3}}$, the ice slides along the inner walls without friction. We place the tube on a plane inclined at an angle $\alpha = 30\,^\circ$ such that it can roll without slipping, and its axis of rotation remains horizontal. What will be the acceleration of the tube's *centre of gravity? The tube's radius is $r = 4.0\,\mathrm{cm}$ and its length is $l = 11\,\mathrm{cm}$.*

Lego wondered about the acceleration of rolling.

The ice in the pipe has a total mass of $m_\mathrm{l} = V_\mathrm{l}\rho = \pi r^2 l\rho$. It also moves at the same velocity as the center of mass of the pipe. Let us denote this velocity as v, then the kinetic energy of the ice is

$$E_\mathrm{l} = \frac{1}{2}\pi r^2 l\rho v^2\,.$$

And since there is no friction between it and the pipe, there is nothing to cause it to rotate, meaning we do not need to consider any rotational kinetic energy.

The shell has a mass of $m_\mathrm{p} = S_\mathrm{p}\sigma = 2\pi rl\sigma$. Its center of mass has a velocity v, so its translational kinetic energy is

$$E_\mathrm{pt} = \pi rl\sigma v^2\,.$$

However, in addition to translation, it also rotates, specifically with an angular velocity such that it does not slip relative to the plane, satisfying $\omega = v/r$. Furthermore, since all the mass is located at a distance r from the axis of rotation, the moment of inertia is $J_\mathrm{p} = m_\mathrm{p}r^2$. The rotational kinetic energy of the shell is then

$$E_\mathrm{pr} = \frac{1}{2}J_\mathrm{p}\omega^2 = \pi rl\sigma r^2\frac{v^2}{r^2} = \pi rl\sigma v^2\,,$$

which could have been deduced without using the moment of inertia.

Thus, the total kinetic energy of the pipe and the ice is

$$E_\mathrm{k} = E_\mathrm{l} + E_\mathrm{pt} + E_\mathrm{pr} = \frac{1}{2}\pi rl(r\rho + 4\sigma)v^2\,,$$

and we can treat $\pi rl(r\rho + 4\sigma) = m_\mathrm{ef}$ as an effective inertial mass, while the actual total mass is $m = m_\mathrm{l} + m_\mathrm{p} = \pi rl(r\rho + 2\sigma)$.

The net force acting on the pipe is the component of its weight parallel to the incline, i.e., $F_\mathrm{v} = mg\sin\alpha$. The acceleration is obtained by dividing by the effective inertial mass as

$$a = \frac{F_\mathrm{v}}{m_\mathrm{ef}} = \frac{\pi rl\,(r\rho + 2\sigma)\,g\sin\alpha}{\pi rl(r\rho + 4\sigma)} = g\sin\alpha\frac{r\rho + 2\sigma}{r\rho + 4\sigma} = 4.0\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem 23 ... delicious fuel — 5 points

Mr. Simpson arrived to a czech nuclear power plant for a conference for nuclear safety engineers. During a security check, the czech technicians found that Mr. Simpson had swallowed a nuclear reactor fuel pellet back at his home power plant. Let us assume that the pellet is made from pure uranium oxide UO_2, which is enriched and is consists of $95\,\%$ of the isotope ^{238}U and $5\,\%$ of the isotope ^{235}U. What is the activity of uranium in the pellet, which Mr. Simpson swallowed? The pellet has a cylindrical shape with a diameter $d = 7.5\,\mathrm{mm}$ and height $h = 11\,\mathrm{mm}$. The density of uranium oxide is $\rho_{\mathrm{UO_2}} = 10.97\,\mathrm{g{\cdot}cm^{-3}}$.

Jindra knows that one pellet is harmless.

Molar mass of oxygen is $m_\mathrm{O} = 16.00\,\mathrm{g{\cdot}mol^{-1}}$. The molar mass of isotope ^{238}U is $m_{238} = 238\,\mathrm{g{\cdot}mol^{-1}}$ and that of the isotope ^{235}U is $m_{235} = 235\,\mathrm{g{\cdot}mol^{-1}}$. The half-lives of these two isotopes are $T_{238} = 4.468 \cdot 10^9\,\mathrm{yr} = 1.410 \cdot 10^{17}\,\mathrm{s}$ and $T_{235} = 7.038 \cdot 10^8\,\mathrm{yr} = 2.221 \cdot 10^{16}\,\mathrm{s}$, respectively. The density of uranium oxide UO_2 is $\rho_{\mathrm{UO}_2} = 10.97\,\mathrm{g{\cdot}cm^{-3}}$. The average molar mass of uranium oxide is

$$m_{\mathrm{UO}_2} = 2m_{\mathrm{O}} + (pm_{235} + (1-p)m_{238}) = 269.85\,\mathrm{g{\cdot}mol^{-1}},$$

where $p = 5\,\%$ is the ratio of ^{235}U. The volumetric mass density of the UO_2 molecules in the pellet is

$$n = \frac{\rho_{\mathrm{UO}_2} N_{\mathrm{A}}}{m_{\mathrm{UO}_2}} = 2.448 \cdot 10^{22}\,\mathrm{cm^{-3}},$$

where N_{A} is Avogadro constant. The volumetric mass densities of the atoms of both uranium isotopes are

$$n_{235} = pn, \quad n_{238} = (1-p)n.$$

The amount of UO_2 in one pellet is not big enough to sustain a controlled fission reaction like in a nuclear reactor or an uncontrolled fission reaction like in a nuclear explosion. The only activity in the pellet therefore comes from the decay of both radioactive isotopes ^{235}U and ^{238}U and their products.

Generally, the number of nuclei N in an radioactive isotope follows the relation

$$N = N_0 2^{-\frac{t}{T}} = N_0 \mathrm{e}^{-\ln(2)\frac{t}{T}},$$

where t is time past since the start of the measurement and N_0 is the number of nuclei at time $t = 0$. The activity of the isotope is

$$A = -\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\ln(2)}{T} N_0 \mathrm{e}^{-\ln(2)\frac{t}{T}} = \frac{\ln(2)}{T} N.$$

The total activity of uranium in the pellet is then

$$A = \frac{\ln(2)pnV}{T_{235}} + \frac{\ln(2)(1-p)nV}{T_{238}},$$

where $V = \pi d^2 h/4 = 0.4860\,\mathrm{cm^3}$ is the volume of the pellet. The activity of the uranium nuclei in the stomach of Mr. Simpson is $A = 74.12\,\mathrm{kBq} \doteq 74.1\,\mathrm{kBq}$.

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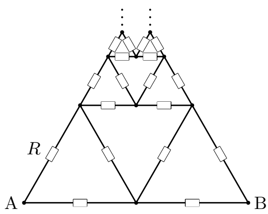
Problem 24 ... infinite resistance 5 points

Determine the resistance of an infinite network of ideal resistors between points A and B. Each resistor in the diagram has the same resistance $R = 1\,\mathrm{k\Omega}$. *Jindra likes triangles.*



Let us denote the point above the point A to the right as C, the point to the left above the point B as D, and the point halfway between points A and B as E. If we remove points A, B, and E and their connections, the resistance R_{tot} between points C and D would be the same as that between points A and B due to symmetry. We use this to calculate the resistance R_{tot} between points A and B.

We will replace the arrangement of the resistors in the triangles ACE and BDE with a star arrangement. Since all resistors have the same resistance R, each leg of the star has the same resistance $R/3$. The resistance between points A and B is

$$R_{\mathrm{tot}} = \frac{R}{3} + \frac{\frac{2R}{3}\left(\frac{2R}{3} + R_{\mathrm{tot}}\right)}{\frac{4R}{3} + R_{\mathrm{tot}}} + \frac{R}{3}\,,$$

$$R_{\mathrm{tot}}\left(\frac{4R}{3} + R_{\mathrm{tot}}\right) = \frac{2R}{3}\left(\frac{4R}{3} + R_{\mathrm{tot}}\right) + \frac{2R}{3}\left(\frac{2R}{3} + R_{\mathrm{tot}}\right)\,,$$

$$R_{\mathrm{tot}}^2 + \frac{4R}{3}R_{\mathrm{tot}} = \frac{8R^2}{9} + \frac{2R}{3}R_{\mathrm{tot}} + \frac{4R^2}{9} + \frac{2R}{3}R_{\mathrm{tot}}\,,$$

$$0 = R_{\mathrm{tot}}^2 - \frac{12R^2}{9}\,,$$

$$R_{\mathrm{tot}} = \frac{2\sqrt{3}}{3}R = 1.154\,701\,\mathrm{k\Omega}\,.$$

The total resistance of this infinite network of resistors is $1.154\,701\,\mathrm{k\Omega}$.

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Problem 25 ... live broadcast 5 points

We are recording a screen with a diagonal of $u = 23\,\mathrm{inch}$ and a resolution of $1\,920\,\mathrm{px} \times 1080\,\mathrm{px}$ from a distance of $d = 1\,\mathrm{m}$ using a camera. The camera feed is transmitted live back onto the screen. Assume that the camera captures a square area with a diagonal field of view $\alpha = 50\,^{\circ}$, that the camera image is adjusted to fit the dimensions of the television (i.e., the originally square image is stretched to fill the entire screen), and that the camera is positioned along the symmetry axis of the screen. Determine how many times we will see the screen displayed within its own image, given that the smallest discernible image of the screen must have an area of at least $100\,\mathrm{px}$ to be distinguishable. *Jarda got a new camera.*

If the camera has a field of view angle α, then in the plane of the television screen, it captures a square area of side length

$$s_{\mathrm{k}} = \left(\sqrt{2}\tan\left(\frac{\alpha}{2}\right)d\right) = 0.659\,\mathrm{m}\,.$$

Knowing the diagonal length and the aspect ratio of the television, we determine its width as

$$s_{\mathrm{t}} = \frac{16}{\sqrt{16^2 + 9^2}}u = 0.509\,\mathrm{m}\,,$$

and its height as

$$v_{\mathrm{t}} = \frac{9}{\sqrt{16^2 + 9^2}}u = 0.286\,\mathrm{m}\,,$$

with the substitution $u = 23\,\mathrm{inch} = 58.42\,\mathrm{cm}$.

The ratio of the television's side length to the width of the area captured by the camera determines how much each side shrinks when projected onto the television. This means that the sides of the screen in the first image shrink to

$$s_{\mathrm{t}} \to s_{\mathrm{t}}\frac{s_{\mathrm{t}}}{s_{\mathrm{k}}} \quad \text{and} \quad v_{\mathrm{t}} \to v_{\mathrm{t}}\frac{v_{\mathrm{t}}}{s_{\mathrm{k}}}\,.$$

The television image, thus scaled down, becomes the new reference. We proceed similarly to obtain a geometric series, where for the number of pixels displaying the n-th image of the television, we have

$$R_n = R_\mathrm{t}\left(\frac{v_\mathrm{t}s_\mathrm{t}}{{s_\mathrm{k}}^2}\right)^n ,$$

where R_t is the total number of pixels on the television, which is

$$R_\mathrm{t} = 1\,920 \times 1\,080 = 2\,073\,600\,\mathrm{px} .$$

By solving the inequality

$$100 < R_\mathrm{t}\left(\frac{v_\mathrm{t}s_\mathrm{t}}{{s_\mathrm{k}}^2}\right)^n ,$$

we find that $n < 9.09$. We need to consider the largest integer less than or equal to this value, so $n_\mathrm{max} = 9$. We can still see the ninth television image, but not the tenth.

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Problem 26 ... periodic deflection — 5 points

Electrons accelerated by a voltage of $1.0\,\mathrm{kV}$ *pass through the center of a deflection plate capacitor which is* $3.0\,\mathrm{cm}$ *long. An alternating voltage of amplitude* $40\,\mathrm{V}$ *and frequency* $50\,\mathrm{Hz}$ *is connected to it. The electrons hit the screen of an oscilloscope located at* $40\,\mathrm{cm}$ *from the end of the capacitor. What is the maximum speed of the beam's trace on the screen? The distance between the plates of the capacitor is* $5.0\,\mathrm{mm}$*. Neglect the electric field outside of it.*

Jarda admires modern flat screens.

Let us calculate the velocity of the incoming electrons. If they were accelerated by a voltage $V = 1.0\,\mathrm{kV}$, then their energy is Ve. Comparing this with their kinetic energy $m_\mathrm{e}v_x^2/2$, we obtain their velocity as

$$v_x = \sqrt{\frac{2Ve}{m_\mathrm{e}}} \doteq 19 \cdot 10^6\,\mathrm{m{\cdot}s^{-1}} ,$$

where m_e is the mass of an electron and e is the elementary charge. The electrons will thus pass through the capacitor in a time

$$\tau = \frac{l_1}{v_x} \doteq 1.6 \cdot 10^{-9}\,\mathrm{s} ,$$

where $l_1 = 3.0\,\mathrm{cm}$ is the length of the capacitor. Therefore, the electrons remain in the capacitor for only a very short time.

During this time, they are subjected to an electric force perpendicular to the capacitor plates. The electric field between the plates is

$$E = \frac{U(t)}{d} = \frac{U_0}{d}\sin\omega t ,$$

where $U(t)$ is the voltage across the capacitor plates, $d = 5.0\,\mathrm{mm}$ is the distance between the plates, $U_0 = 40\,\mathrm{V}$ is the amplitude of this voltage, and $\omega = 2\pi f = 314\,\mathrm{s^{-1}}$ is the angular frequency of the voltage oscillation.

However, this field changes much more slowly than the electron's transit time. We can therefore assume that the field's magnitude remains constant during the electron's time in the capacitor and that it exerts a constant force on the electron. This force is

$$F = -Ee = -\frac{U_0 e}{d}\sin \omega t\,.$$

Thus, the electron moves perpendicular to the plates with an acceleration $a_y = F/m_\mathrm{e}$ during the time τ. In this direction, when passing through the capacitor it gains the velocity

$$v_y = a_y \tau = -\frac{U_0 e}{d m_\mathrm{e}}\tau \sin \omega t\,.$$

The direction of its flight after leaving the capacitor is determined by the ratio v_y to v_x. From the similarity of triangles, we determine the distance y between the symmetry plane and the point of impact on the screen as

$$\frac{y}{l_2} = \frac{v_y}{v_x} \quad \Rightarrow \quad y = l_2 \frac{v_y}{v_x} = -l_1 l_2 \frac{1}{v_x^2}\frac{U_0 e}{d m_\mathrm{e}}\sin \omega t = -\frac{l_1 l_2}{d}\frac{U_0}{2V}\sin \omega t\,.$$

Here, $l_2 = 40\,\mathrm{cm}$ is the distance from the end of the capacitor to the screen.

We see that this position depends periodically on time. It represents the position of the beam's trace on the screen. To find the speed at which this trace moves, we calculate its derivative as

$$v_\mathrm{s} = \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{l_1 l_2}{d}\frac{U_0}{2V}\omega \cos \omega t\,.$$

The position changes with this velocity over time. The maximum value occurs whenever $\cos \omega t = 1$, and its magnitude is

$$v_\mathrm{s,max} = \pi f \frac{l_1 l_2}{d}\frac{U_0}{V} = 15\,\mathrm{m{\cdot}s^{-1}}\,,$$

where we substituted $\omega = 2\pi f$ for the angular frequency.

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Problem 27 ... use crosswalks 5 points

Jarda decided to cross a road of width $5.0\,\mathrm{m}$ *when a car approached him in the middle of the far lane at* $50\,\mathrm{km{\cdot}h^{-1}}$. *The car was* $1.5\,\mathrm{m}$ *wide, driving in the middle of its lane, and had no intention of braking. Jarda ran across the road in front of the car at a speed of* $3.0\,\mathrm{m{\cdot}s^{-1}}$. *How close to Jarda (along the road) could the car have been when he decided to step into the road if the car did not hit him? Jarda was running across the road in a straight line.*

Jarda is not always using crosswalks...

Jarda enters the road with a speed $u = 3\,\mathrm{m{\cdot}s^{-1}}$ at an angle α to the normal to the roadside. To avoid being hit by the car, he must avoid the corner of the right front bumper of the car. This corner is moving at a speed $v = 50\,\mathrm{km{\cdot}h^{-1}}$ at a distance $d = 4.5\,\mathrm{m}$ (since the lane is $2.5\,\mathrm{m}$ wide and the car is driving in the middle, its edge is $0.5\,\mathrm{m}$ away from the far side of the road).

Jarda travels the distance d in the time

$$t = \frac{d}{u\cos\alpha}\,.$$

The car travels $L' = vt$ in this time. Let us denote L the distance we are solving for. We can arrange the equation as

$$L = L' - d\tan\alpha = d\left(\frac{v}{u\cos\alpha} - \tan\alpha\right) = \frac{d}{\cos\alpha}\left(\frac{v}{u} - \sin\alpha\right) .$$

We plot this function $L(\alpha)$ in some graphics program (e.g. Geogebra) and simply deduce that the minimum value is approximately $20\,\mathrm{m}$. This is not a very large distance, it is definitely safer to give yourself a larger margin when crossing a road. We could also use derivation and calculate the result.

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Problem 28 ... shot at a better future 5 points

From a cannon firmly attached to the ground, situated at the equator, we shoot a ball weighing $5\,\mathrm{kg}$ at a 45° angle, which lands $50\,\mathrm{m}$ away from the cannon. How much would the sidereal day shorten if the Earth rotated with the same angular velocity as the ball immediately after the shot? Assume the shot is fired westward, and approximate the Earth as a homogeneous sphere. Consider only the gravitational effects of Earth.

Monika is aiming to make fun of the problem-checkers.

We will solve the problem using the law of conservation of angular momentum. This law states that the total angular momentum of an isolated system must be conserved, meaning

$$L_{Z2} = L_{Z1} + L_{\mathrm{s}} , \qquad (2)$$

where L_{Z1} is Earth's initial angular momentum, L_{Z2} is Earth's angular momentum immediately after the shot, and L_{s} the angular momentum of the projectile. This can be expressed in vector form as

$$\mathbf{L}_{\mathrm{s}} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} ,$$

where $m = 5\,\mathrm{kg}$ is the mass of the projectile, $\mathbf{v}$ the velocity vector of the projectile, and $\mathbf{r}$ is the position vector (directed from Earth's center to the cannon). Since we are only interested in the magnitude of angular momentum, we write

$$L_{\mathrm{s}} = mR_{\oplus}v_x ,$$

where $R_{\oplus} = 6.378 \cdot 10^6$ m is Earth's equatorial radius and v_x is the horizontal component of the projectile's velocity. Because the projectile is fired at an angle of 45°, the horizontal component is equal to the vertical component. Given that the projectile travels a horizontal distance $x = 50$ m under gravitational acceleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$, we use the projectile motion equations at the point of impact

$$x = v_x t ,$$
$$0 = v_x t - \frac{1}{2}gt^2 .$$

From the first equation, we solve for t, substitute into the second equation, and solve for v_x:

$$v_x = \sqrt{\frac{gx}{2}} .$$

Returning to equation (2), the angular momentum of a sphere of mass M and radius r, rotating with angular speed ω, is given by

$$L_\circ = \frac{2}{5} M_\oplus R_\oplus^2 \omega_1 \,,$$

where $\omega_1 = 7.292 \times 10^{-5}\,\mathrm{s^{-1}}$ is Earth's angular velocity (one rotation in a sidereal day of 23 hours, 56 minutes, and 4 seconds). After the shot, Earth's angular velocity becomes $\omega_2 = \omega_1 + \Delta\omega$. Substituting these into equation (2)

$$\frac{2}{5} M_\oplus R_\oplus^2 (\omega_1 + \Delta\omega) = \frac{2}{5} M_\oplus R_\oplus^2 \omega_1 + m v_x R_\oplus \,,$$

we solve for $\Delta\omega$

$$\Delta\omega = \frac{5 m v_x}{2 M_\oplus R_\oplus} \,.$$

Now, we find the change in the length of the day ΔT caused by $\Delta\omega$. The relationship between angular velocity and the period is

$$\Delta T = \frac{2\pi}{\omega_1} - \frac{2\pi}{\omega_1 + \Delta\omega} = \frac{2\pi}{\omega_1}\left(1 - \frac{1}{1 + \frac{\Delta\omega}{\omega_1}}\right) \overset{(*)}{\approx} 2\pi \frac{\Delta\omega}{\omega_1^2} = \frac{5\pi m}{M_\oplus R_\oplus \omega_1^2} v_x = \frac{5\pi m}{M_\oplus R_\oplus \omega_1^2} \sqrt{\frac{gx}{2}} \,,$$

where in step $(*)$, we used the approximation $1/(1+x) \approx 1 - x$ for small x (which is valid here, as verified after substitution).

Substituting numerical values we get

$$\Delta T \doteq 6.1 \times 10^{-21}\,\mathrm{s} \,.$$

This shows that ΔT is negligible compared to the length of the day, confirming that the assumption $\Delta\omega/\omega_1 \ll 1$ was valid.

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Problem 29 ... hanging phone 5 points

Petr's phone, weighing $m = 200\,\mathrm{g}$, fell off a shelf while connected to a charger plugged into a wall socket. The charger cable consists of two copper wires with a circular cross-section, each of length $l_0 = 1.00\,\mathrm{m}$ and diameter $d_0 = 1.50\,\mathrm{mm}$. The phone remained hanging on it after the fall. By how many nanoohms did the resistance of the charger change? Assume that the charger forms a series circuit with the phone and the power source in the socket. The Young's modulus of elasticity for copper is $E = 110\,\mathrm{GPa}$, its Poisson's ratio is $\nu = 0.340$, and its specific electrical resistivity is $\rho = 16.78\,\mathrm{n\Omega{\cdot}m}$. Do not consider the effect of the plastic insulation of the cable. *Petr's phone fell to the ground (again).*

First, let us calculate how the dimensions of the wires change. The hanging phone exerts a force on the charger, $W = mg$. Since the cross-sections of the wires are circular, the areas are given as $S_0 = \pi d_0^2/4$, and the stress σ on each wire is

$$\sigma = \frac{mg}{2S_0} = \frac{2mg}{\pi d^2} \doteq 555.1\,\mathrm{kPa} \,,$$

where the weight W is divided by two because the phone is suspended by two wires.
The new length of the wire can be calculated using Hooke's law as

$$l = l_0 \left(1 + \frac{\sigma}{E}\right) \doteq 1.000\,005\,\mathrm{m}\,.$$

However, the wire also narrows due to the stretching. The Poisson's ratio for copper is $\nu = 0.340$, so the new diameter is given by

$$d = d_0 \left(1 - \frac{\nu\sigma}{E}\right)\,.$$

To find the resistance of the charger, we use the formula involving specific resistivity

$$R = \rho \frac{l}{S}\,.$$

For the difference between the original and the new resistance, we have

$$R - R_0 = 2\rho \left(\frac{l}{S} - \frac{l_0}{S_0}\right)\,,$$

where the coefficient 2 in front of the term on the right accounts for the fact that we are calculating the change in resistance for two identical wires connected in series. Substituting, we get

$$R - R_0 = \frac{2\rho l_0}{S_0} \left(\frac{1 + \frac{\sigma}{E}}{\left(1 - \frac{\nu\sigma}{E}\right)^2} - 1\right)\,.$$

To simplify this expression further, we expand the coefficient in the first term of the parentheses into a Taylor series

$$\frac{1}{\left(1 - \frac{\nu\sigma}{E}\right)^2} \Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + o(x^2)\,.$$

Substituting the series expansion into the original formula and neglecting the quadratic term in the Taylor expansion, we get

$$R - R_0 = \frac{2\rho l_0 \sigma}{S_0 E} \left(2\nu + 1\right)\,,$$

which, after substituting the numerical values, yields

$$R - R_0 \doteq 161.0\,\mathrm{n\Omega}\,.$$

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Problem 30 ... concentric spheres

5 points

Let there be an infinite number of concentric spherical shells with a uniformly distributed charge. The smallest sphere with a radius $r = 0.1\,\mathrm{mm}$ has a positive charge of $2Q$, where Q is the elementary charge. All the other spheres have a positive charge of Q, and the radius of the n-th sphere is n-times greater than the radius of the $(n-1)$-th sphere. What is the potential on the surface of the smallest sphere, assuming the potential at infinity is zero?

Monča found a potential use of some approximation.

The potential of the electric field on a sphere behaves the same way as a potential around a point charge with the same charge as the sphere. It can therefore be expressed as

$$\varphi = k\frac{q}{r}\,,$$

where q is the charge and $k = 1/(4\pi\varepsilon)$ is a constant describing the medium. According to the principle of superposition, the electric fields of the individual spheres do not influence each other. Thus, the total potential on the surface of the smallest sphere will be given by the sum of the potentials of all the individual spheres – we need to sum over infinitely many layers. If we expand the first few terms of the sum, we observe a certain pattern:

$$\begin{aligned}\varphi_1 &= k\frac{2Q}{r}\,,\\ \varphi_2 &= k\frac{Q}{2r}\,,\\ \varphi_3 &= k\frac{Q}{2\cdot 3r} = k\frac{Q}{3!r}\,,\\ \varphi_4 &= k\frac{Q}{2\cdot 3\cdot 4r} = k\frac{Q}{4!r}\,.\end{aligned}$$

It is not difficult to verify that this trend will continue, and for any n-th layer (for $n \geq 2$), the expression will hold:

$$\varphi_n = k\frac{Q}{n!r}\,.$$

Now, we can factor out the constant kQ/r from the entire infinite sum, yielding:

$$\varphi_{\mathrm{total}} = k\frac{Q}{r}\left(2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots\right)\,.$$

The infinite series in parentheses is the expansion of Euler's number. The solution can therefore be written as:

$$\varphi_{\mathrm{total}} = k\frac{Qe}{r}\,.$$

After substituting numerical values, we find:

$$\varphi_{\mathrm{total}} = k\frac{Qe}{r} = 9\cdot 10^9\,\frac{1.602\cdot 10^{-19}e}{1\cdot 10^{-4}} = 3.9\cdot 10^{-5}\,\mathrm{V}\,.$$

The potential on the surface of the smallest sphere is $\varphi_{\mathrm{total}} = 3.9\cdot 10^{-5}\,\mathrm{V}$.

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Problem 31 ... measured gas 5 points

Let us completly submerge a measuring cylinder into water, turn it upside down and pull it out so that the part above the water's surface is filled with water. Then, we introduce a tube beneath it, through which flows argon from a gas cylinder. The gas bubbles up into the measuring cylinder, creating a gas pocket at the top. Once the argon stabilizes and the water level inside and outside of the cylinder equilizes, we measure that the volume of the gas is $0.85\,\mathrm{l}$. *What is the mass of argon in the measuring cylinder? The temperature of both the water and the inside of the cylinder is* $81\,\mathrm{^\circ C}$ *and the external pressure is atmospheric.*

Jarda is again trying to catch you off guard this year.

Once the levels equalise, the pressure inside is the same as outside, i.e. atmospheric. Therefore inside

$$n = \frac{pV}{RT}$$

moles of gas. However, the experiment takes place at a relatively high temperature when the saturation vapour pressure is already significant. Water molecules evaporate from the surface into the volume of the cylinder, but in equilibrium, the same amount condenses back. The total pressure p_a inside of the cylinder is thus given by the sum of the pressure of the water molecules $p_{\mathrm{H_2O}}$ and argon p_Ar, and similarly for the sum of all particles.

The pressure of saturated water vapour at $81\,\mathrm{^\circ C}$ je asi $p_{\mathrm{H_2O}} = 0.05\,\mathrm{MPa}$,[2] which is about half the atmospheric pressure. The partial pressure of argon and the number of moles of argon particles in the cylinder is therefore

$$p_\mathrm{Ar} = p_\mathrm{a} - p_{\mathrm{H_2O}} \doteq 50\,\mathrm{kPa}, \quad n_\mathrm{Ar} = \frac{p_\mathrm{Ar} V}{RT} \doteq 0.014\,\mathrm{mol}\,.$$

The molar mass of argon is $M_\mathrm{Ar} \doteq 40\,\mathrm{g{\cdot}mol^{-1}}$, its mass in a cylinder is hence

$$m_\mathrm{Ar} = M_\mathrm{Ar} n_\mathrm{Ar} = M_\mathrm{Ar} \frac{(p_\mathrm{a} - p_{\mathrm{H_2O}})V}{RT} \doteq 0.58\,\mathrm{g}\,.$$

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Problem 32 ... weather watching 5 points

A meteorological satellite continuously captures images of the Earth's surface and transmits the data back to Earth at a constant frequency f. *Jindra first picked up the signal from the satellite on his antenna when the satellite climbed just above the horizon. The frequency was* $f_\mathrm{max} = 137.915\,51\,\mathrm{MHz}$. *The satellite subsequently climbed further above the horizon and reached its maximum altitude. The last signal Jindra received was just before the satellite went below the horizon. The frequency at that time was* $f_\mathrm{min} = 137.909\,49\,\mathrm{MHz}$. *What is the radius of the satellite's orbit?*

Assume that the satellite orbits the Earth in a circular orbit. Neglect the rotation of the Earth and its effect on the frequency shift.

Jindra was at the Expo.

The frequency shift occurs due to the Doppler effect. In the leading order, the Doppler effect is influenced only by the projection of the velocity vector onto a line connecting the satellite and

[2] See e.g. `https://www.tzb-info.cz/tabulky-a-vypocty/9-vlastnosti-syte-vodni-pary-pri-danem-tlaku`.

the observer. The satellite orbits the Earth in a circular trajectory with a for-now-unknown radius R. Its speed is constant and given by

$$v = \sqrt{\frac{GM}{R}},$$

where G is the gravitational constant and $M = 5.97 \cdot 10^{24}\,\mathrm{kg}$ is the mass of the central body, in this case, Earth.

Since the satellite flew directly overhead of Jindra, its velocity projection during its horizon crossing was

$$u = \sqrt{\frac{GM}{R}}\frac{R_\oplus}{R},$$

where $R_\oplus$ is the Earth's radius.

When the satellite rose above the horizon, the observer measured a higher frequency

$$f_{\mathrm{max}} = \frac{c}{c-u}f,$$

because at that moment, the distance between the observer and the satellite was decreasing. At the satellite's setting below the horizon, the observer measured a lower frequency

$$f_{\mathrm{min}} = \frac{c}{c+u}f,$$

because at that moment, the satellite was moving away.

These two equations can be combined into one

$$\begin{aligned} f_{\mathrm{max}}(c-u) &= f_{\mathrm{min}}(c+u),\\ u &= c\frac{f_{\mathrm{max}} - f_{\mathrm{min}}}{f_{\mathrm{max}} + f_{\mathrm{min}}},\\ \sqrt{GMR_\oplus^2}\frac{1}{R^{3/2}} &= c\frac{f_{\mathrm{max}} - f_{\mathrm{min}}}{f_{\mathrm{max}} + f_{\mathrm{min}}},\\ R &= \sqrt[3]{\frac{GMR_\oplus^2}{c^2}\left(\frac{f_{\mathrm{max}} + f_{\mathrm{min}}}{f_{\mathrm{max}} - f_{\mathrm{min}}}\right)^2} =\\ &= 7\,229\,\mathrm{km} \doteq 7\,230\,\mathrm{km}. \end{aligned}$$

The satellite orbits the Earth in a circular trajectory with a radius $R = 7\,230\,\mathrm{km}$.

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Problem 33 ... overpriced taxicab 6 points

During the taxi ride, Monča realized it could not be profitable for any taxicab if the customer used its services for too long. However, she let her imagination run loose and fantasized about a taxi that gradually increased its fare during the ride. Let us imagine a hypothetical taxi operating as follows: for the first hour of the ride, it charges a fixed final price, and then charges this amount for the following 59 minutes of the ride. Then, the driving time, which *costs this price, continues to decrease as a geometric sequence. Furthermore, this taxi travels along segments of length s_0. The first segment takes 1 hour to traverse, the second one in 1 hour and 1 minute, and this travel time for the segment of length s_0 further increases as a geometric sequence.*

Consider the shortest possible trip for which we would pay an infinite sum. Determine the overall average speed of such a trip if the speed of the taxi is constant in each segment, with the taxi traveling at $30\,\mathrm{m\cdot s^{-1}}$ in the first segment. *Monča got scammed in a taxicab.*

To calculate the average speed, we need to determine the total time t and the total distance s of the taxi ride.

First, let us calculate the total time. The duration of each segment that costs the same amount decreases as a geometric sequence. We can calculate its quotient using the first two terms

$$q = \frac{t_2}{t_1}\,,$$

where $t_1 = 60\,\mathrm{min}$ a $t_2 = 59\,\mathrm{min}$.

To ensure the taxi ride costs an infinite sum, we must pay the finite rate infinitely many times. Thus, we calculate the minimum travel time t by summing an infinite series with a common ratio q and the first term t_1. We get

$$t = t_1 \sum_{i=0}^{\infty} q^i = \frac{t_1}{1-q}\,. \tag{3}$$

Now, let us move on to the calculation of the total distance. The taxi covered the first segment at speed $v_1 = 30\,\mathrm{m\cdot s^{-1}}$ for a time t_1. The length of each segment is therefore $s_0 = v_1 t_1$. The total distance traveled increases with number of segments as geometric sequence. We can get its quotient as

$$r = \frac{\tau_2}{\tau_1}\,,$$

where $\tau_1 = 60\,\mathrm{min}$ a $\tau_2 = 61\,\mathrm{min}$. The duration of the ride across n segments can be expressed as

$$\tau = \tau_1 \frac{r^n - 1}{r-1}\,. \tag{4}$$

By expressing the number of segments n from (4), we can substitute τ with the real duration t with respect to (3)

$$n = \log_r \left[1 + \frac{(r-1)t}{\tau_1}\right] = \frac{1}{\ln r} \ln\left(1 - \frac{r-1}{q-1}\frac{t_1}{\tau_1}\right) \doteq 41.9\,.$$

Our taxi went through 41 whole segments with length s_0 and a majority of another segment. On this segment the taxi traveled for time $t - \tau$ at a speed

$$v_{42} = \frac{s_0}{\tau_{42}} = \frac{v_1}{r^{41}}\,,$$

which means that that it covered total distance

$$s = 41 s_0 + v_{42}(t-\tau) = v_1\left(41 t_1 + \frac{t-\tau}{r^{41}}\right)\,.$$

Finally, for the average speed over the total distance traveled, we get

$$
\begin{aligned}
v &= \frac{s}{t} \\
&= \frac{v_1}{t}\left(41t_1 + \frac{t-\tau}{r^{41}}\right) \\
&= v_1\left[41(1-q) + \frac{1}{r^{41}}\left(1 + \frac{\tau_1(r^{41}-1)(q-1)}{t_1(r-1)}\right)\right] \\
&\doteq 75.5\,\mathrm{km{\cdot}h^{-1}}\,.
\end{aligned}
$$

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Problem 34 ... pressurized bubble in oil — 5 points

What additional force would we need to apply on a piston, which leads into a tank containing a small air bubble in oil, to reduce its radius to half? The piston has a cross-sectional area of $S = 4.20\,\mathrm{cm^2}$. For simplicity, assume that the air in the bubble was initially at normal conditions and its initial radius was $r = 0.420\,\mathrm{mm}$. The surface tension of the oil is $\sigma = 3.42\cdot 10^{-4}\,\mathrm{N{\cdot}m^{-1}}$, the oil conducts heat well, and its temperature does not change significantly during the process.

Karel was thinking about an undesirable bubble.

We do not have the height of the piston or similar details, so we ignore the effects of hydrostatic pressure (these would likely cancel out unless the bubble moved). According to Pascal's law, the pressure in the oil will then be uniform and equal to the pressure from the piston $p = F/S$. If we denote the initial pressure in the oil as p_1 and the pressure needed to compress the bubble to half its original radius as p_2, then (since the piston area remains constant) the additional required force is given by

$$\Delta F = S(p_2 - p_1)\,.$$

However, due to the surface tension of the oil, the pressure inside the bubble will differ. Specifically, the capillary pressure of the bubble is

$$\Delta p_\mathrm{k} = \frac{2\sigma}{R}\,,$$

where σ is the surface tension of the liquid forming the bubble, and R is the bubble's radius. Note: you may find a formula online with a coefficient of 4 instead of 2, indicating twice the capillary pressure. However, this is for a bubble in the sense of air-film-air, which has two surfaces (inner and outer), thus exerting twice the pressure compared to a "bubble" inside a liquid (only pushed by one surface).

We know that the initial pressure inside the bubble was p_A. Given the initial radius and the corresponding capillary pressure, we can calculate the initial pressure in the oil as follows

$$p_1 = p_{in1} - \Delta p_{k1} = p_\mathrm{A} - \frac{2\sigma}{r}\,.$$

We also know the final radius of the bubble, allowing us to determine the final capillary pressure. Now, we only need to find the pressure required to compress the bubble. The problem indicates that the oil conducts heat effectively and maintains a constant temperature.

Consequently, the temperature of the bubble will also remain constant. Furthermore, we can reasonably assume that no air escapes from the bubble, leading to an isothermal process. In an isothermal process, pressure and volume are inversely proportional. Compressing the bubble to half its radius means its volume decreases by a factor of 8, causing the pressure to increase by a factor of 8

$$
\begin{aligned}
p_{\mathrm{in}} &= \frac{C}{V} = \frac{C}{\frac{4}{3}\pi R^3} = \frac{C_2}{R^3}\,,\\
p_{\mathrm{in1}} &= \frac{C_2}{r^3} = p_{\mathrm{A}}\,,\\
p_{\mathrm{in2}} &= \frac{C_2}{(r/2)^3} = 8\frac{C_2}{r^3} = 8p_{\mathrm{in1}} = 8p_{\mathrm{A}}\,.
\end{aligned}
$$

The required pressure in the oil is

$$p_2 = p_{\mathrm{in2}} - \Delta p_{\mathrm{k2}} = 8p_{\mathrm{A}} - \frac{2\sigma}{r/2} = 8p_{\mathrm{A}} - \frac{4\sigma}{r}\,,$$

which we can substitute into the expression for the additional force

$$\Delta F = S(p_2 - p_1) = S\left(\left(8p_{\mathrm{A}} - \frac{4\sigma}{r}\right) - \left(p_{\mathrm{A}} - \frac{2\sigma}{r}\right)\right) = S\left(7p_{\mathrm{A}} - \frac{2\sigma}{r}\right) \doteq 298\,\mathrm{N}\,.$$

We can observe that the contribution from the capillary pressure is only $\Delta F_{\mathrm{k}} = 2S\sigma/r \doteq 0.7\,\mathrm{mN}$. Thus, we could have reasonably ignored the surface tension.

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Problem 35 ... dramatic escape 6 points

Marek is running away from his responsibilities across a bridge of height $H = 4.0\,\mathrm{m}$ but realizes he is surrounded. He decides to jump onto a boat of length $L = 8.0\,\mathrm{m}$, whose bow is just emerging from under the bridge and is moving forward at a speed of $v_{\mathrm{l}} = 5.0\,\mathrm{m\cdot s^{-1}}$. What is the difference between the maximum and minimum angles α at which Marek can jump from the bridge at a speed of $v = 8.0\,\mathrm{m\cdot s^{-1}}$ to land on the boat? Assume that Marek jumps upwards and neglect the height of the boat.

Marek is an outlaw. Jarda wants him to proofread this year's series.

We introduce a coordinate system originating at Marek's initial position. The x-axis is parallel to the boat's path, and the y-axis is perpendicular to it. The entire system moves in the same direction and at the same speed as the boat. In this coordinate system, Marek's position changes as follows

$$
\begin{aligned}
x &= (v\cos\alpha - v_{\mathrm{l}})\,t\,,\\
y &= vt\sin\alpha - \frac{1}{2}gt^2\,,
\end{aligned}
$$

where α is the angle at which Marek jumps relative to the x-axis and t is the time from the moment he jumps. Expressing t from the first equation and substituting it into the second, we obtain

$$y = \frac{xv\sin\alpha}{v\cos\alpha - v_{\mathrm{l}}} - \frac{gx^2}{2(v\cos\alpha - v_{\mathrm{l}})^2}\,,$$

which we can rewrite as

$$y = \frac{xv\sqrt{1-\cos^2\alpha}}{v\cos\alpha - v_l} - \frac{gx^2}{2(v\cos\alpha - v_l)^2}\,.$$

We introduce the substitution

$$\cos\alpha = k\,,$$
$$y = \frac{xv\sqrt{1-k^2}}{vk - v_l} - \frac{gx^2}{2(vk - v_l)^2}.$$

We now have an equation that must be solved numerically (as terms up to k^4 appear after expansion). We substitute the values $y = -H$ and $x = -L$ and use a numerical solver to find the resulting values for k (and subsequently $\cos\alpha$)

$$\cos\alpha_{\max} = 0.13012 \quad\Rightarrow\quad \alpha_{\max} = 82.52\,^\circ\,.$$

Thus, we found the maximum angle at which Marek can jump.

We can easily find the minimum angle from the condition that he must land precisely at the bow. Thus, his velocity in the x-direction in our coordinate system must be zero

$$\cos\alpha_{\min} = \frac{v_l}{v} \quad\Rightarrow\quad \alpha_{\min} = 51.32\,^\circ\,.$$

The range of angles at which Marek can jump is therefore

$$\Delta\alpha \doteq 31.2\,^\circ.$$

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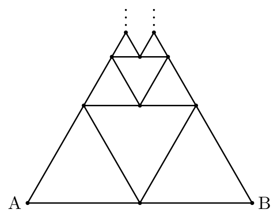
Problem 36 ... endless resistance II 6 points

Determine the resistance of an infinite network of wires between points A and B. The triangles forming the network are equilateral, and the distance between the points A and B is $a = 1.000\,000\,\mathrm{m}$. All wires are of the same type and have a resistivity of $\lambda = 1.000\,000\,\Omega\cdot\mathrm{m}^{-1}$. Jindra is getting tangled up in the wires.



We label the point to the upper right of point A as C, the point to the upper left of point B as D, and the midpoint between A and B as E. If we were to remove points A, B, E, and their connections, the resistance between points C and D would be half of the total resistance R_{tot} between points A and B. That is because all wires have half the length and thus half the resistance in the triangle between C and D. We use this fact to calculate the resistance between A and B.

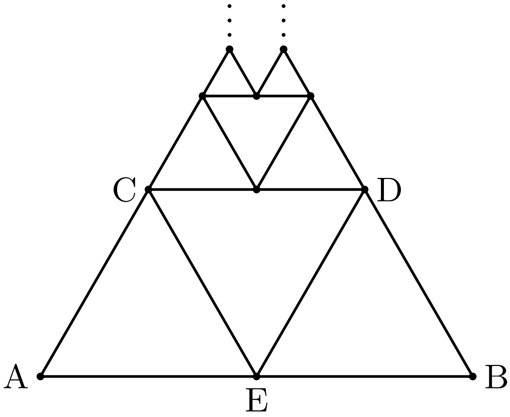


Figure 2: Highlighting points C, D, and E.

First, we replace the arrangement of resistors in the triangles ACE and BDE with a star configuration. Given that all resistors have the same resistance $R = \lambda a/2$, each limb of the star has the same resistance $R/3$. The resistance between points A and B is

$$R_{\mathrm{tot}} = \frac{R}{3} + \frac{\frac{2R}{3}\left(\frac{2R}{3} + \frac{R_{\mathrm{tot}}}{2}\right)}{\frac{4R}{3} + \frac{R_{\mathrm{tot}}}{2}} + \frac{R}{3}\,,$$

$$R_{\mathrm{tot}}\left(\frac{4R}{3} + \frac{R_{\mathrm{tot}}}{2}\right) = \frac{2R}{3}\left(\frac{4R}{3} + \frac{R_{\mathrm{tot}}}{2}\right) + \frac{2R}{3}\left(\frac{2R}{3} + \frac{R_{\mathrm{tot}}}{2}\right)\,,$$

$$\frac{R_{\mathrm{tot}}^2}{2} + \frac{4R}{3}R_{\mathrm{tot}} = \frac{8R^2}{9} + \frac{R}{3}R_{\mathrm{tot}} + \frac{4R^2}{9} + \frac{R}{3}R_{\mathrm{tot}}\,,$$

$$3R_{\mathrm{tot}}^2 + 4RR_{\mathrm{tot}} - 8R^2 = 0\,,$$

$$R_{\mathrm{tot}} = \frac{-4R \pm \sqrt{16R^2 + 96R^2}}{6} = \frac{2}{3}R\left(-1 \pm \sqrt{7}\right)\,.$$

Only the positive solution is physically meaningful. Substituting for R, we get

$$R_{\mathrm{tot}} = \left(\sqrt{7} - 1\right)\frac{2R}{3} = \left(\sqrt{7} - 1\right)\frac{\lambda a}{3} \doteq 0.548\,583\,8\,\Omega.$$

Thus, the resistance of this infinite network between points A and B is approximately $0.548\,583\,8\,\Omega$.

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Problem 37 ... dipole on a spring 5 points

Consider two charged particles which together form an electric dipole with a moment $p_0 = 2.20 \cdot 10^{-25}\,\mathrm{C \cdot m}$. *The system is in its energy minimum, where the particles are separated by a distance of* $l_0 = 3.10\,\mathrm{nm}$. *When displaced, they behave as they were connected by spring of stiffness* $k = 4.80\,\mathrm{mN{\cdot}m^{-1}}$. *What would be the dipole moment if the dipole was moving at* *a speed of* $v = 28.0\,\mathrm{km\cdot s^{-1}}$ *perpendicular to the direction of its dipole moment in a homogeneous magnetic field with an induction of* $B = 250\,\mathrm{mT}$*, which is perpendicular to both the dipole's velocity vector and the direction of the dipole moment? Provide the highest possible value.*

Jarda changes in the magnetic field.

For the value of the electric dipole outside of the magnetic field, we consider the relation

$$p_0 = ql_0\,,$$

where $q = 7.1 \cdot 10^{-17}\,\mathrm{C}$ is the charge of the particles.

In a magnetic field, each charge is also subjected to a magnetic force of magnitude Bvq, where q is the charge of the particle. The sign of the charge differs for the particles, but the velocity vector, magnetic induction, and charge magnitude are the same for both particles. Thus, each particle experiences a force in the opposite direction. From the equality of forces on the spring, we obtain

$$k\left(l_\mathrm{m} - l_0\right) = \pm Bvq\,,$$

where the sign depends on the direction in which the charges are moving relative to the magnetic field. The quantity l_m denotes the distance between the particles moving in the magnetic field. If we adopt a coordinate system where the vector B points along the z-axis, the velocity vector along the x-axis, and in a right-handed Cartesian system the positive charge is on the positive y-coordinate, the sign in the equation is $-$ (the magnetic force pushes the charges together). In the opposite case, the sign is $+$ (the magnetic force acts against the attractive electric force).

The task now is to find the value of the product $p_\mathrm{m} = ql_\mathrm{m}$. To determine the maximum possible value, we require the magnetic force to push the charges apart, maximizing their distance. Therefore, we consider only the $+$ sign in the equation.

Expressing l_m gives

$$l_\mathrm{m} = \frac{Bvq}{k} + l_0\,,$$

which we multiply by the charge q to find the desired dipole moment

$$p_\mathrm{m} = \frac{Bvq^2}{k} + p_0 = \frac{Bvp_0^2}{kl_0^2} + p_0 = 2.27 \cdot 10^{-25}\,\mathrm{C\cdot m}\,.$$

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Problem 38 ... not-so-warm water 6 points

At a FYKOS trip, Petr stepped into the shower at midnight. When he turned on the water, a fully filled boiler with a volume of $V = 100\,\mathrm{l}$ *started heating its contents. During this process, water flows out to the shower at a rate of* $Q = 0.1\,\mathrm{l\cdot s^{-1}}$*, and cold water flows into the boiler at the same rate from the supply. The temperature of the cold water from the supply is* $T_\mathrm{s} = 20\,\mathrm{^\circ C}$*, which is also the initial temperature of the water in the boiler. How long will it take before Petr can shower with warm water at a temperature of* $T_\mathrm{k} = 40\,\mathrm{^\circ C}$*? The boiler has a power output of* $P = 15\,\mathrm{kW}$*. Assume that the incoming water mixes perfectly with the water in the boiler.*

Petr is not a cold water lover.

In addition to heating with power P, the volume V is effectively cooled by incoming water with a cooling power P_{ch}

$$P_{\mathrm{ch}} = -Q\rho c\left(T - T_{\mathrm{s}}\right) ,$$

where T is the temperature of the water in the boiler, ρ is the density of the water, and c is its specific heat capacity. On the other hand, the boiler is heated with power P.

We can express the heat stored in the boiler as $\tau = V\rho cT$. The change in heat per unit of time is thus directly proportional to the change in temperature as

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = V\rho c\frac{\mathrm{d}T}{\mathrm{d}t} ,$$

which equals the total thermal power. Therefore, we have

$$V\rho c\frac{\mathrm{d}T}{\mathrm{d}t} = P - Q\rho c\left(T - T_{\mathrm{s}}\right) ,$$

which is a differential equation with separable variables for T as a function of time t. Its general solution is

$$T = \frac{1}{Q\rho c}\left(P - \exp\left(-\frac{Q}{V}\left(t + k\right)\right)\right) + T_{\mathrm{s}} .$$

By substituting the initial condition $T(0) = T_{\mathrm{s}}$, we find the constant k

$$c = -\frac{V}{Q}\ln P ,$$

so the particular solution to the equation is

$$T = \frac{P}{Q\rho c}\left(1 - \exp\left(-\frac{Q}{V}t\right)\right) + T_{\mathrm{s}} .$$

The time t_{k} will be attained when the water reaches the desired temperature $T_{\mathrm{k}} = 40\,^{\circ}\mathrm{C}$. Substituting and simplifying, we get

$$t_{\mathrm{k}} = -\frac{V}{Q}\ln\left(1 - \frac{Q\rho c}{P}\left(T_{\mathrm{k}} - T_{\mathrm{s}}\right)\right) .$$

After substituting numerical values, we obtain

$$t_{\mathrm{k}} \doteq 8.0 \cdot 10^{2}\,\mathrm{s} .$$

Thus, Petr will wait more than thirteen minutes for warm water.

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Problem 39 ... hourglass

6 points

Let us consider an hourglass which, instead of sand, contains water. The apex angle is $40\,^\circ$*, and the gap between the chambers has a diameter of* $2.0\,\mathrm{mm}$*. What must be the volume of one chamber, if an initially fully filled chamber empties in* $2\,\mathrm{min}$ *after being flipped? The entire double cone is rotationally symmetrical, and the two chambers are connected by a thin tube along the side so that the pressure of the gases inside is not a factor. Neglect the surface tension of water and assume that the connection between the chambers is small compared to the other dimensions.*

Jarda made use of Anika K's suggestion.

Let us denote the height h of the water level above the opening when the hourglass is in a vertical position. Then, the outflow velocity, according to Torricelli's law, is

$$v = \sqrt{2gh}\,.$$

The volumetric flow rate through an opening of diameter d is then

$$Q = \frac{\pi d^2}{4} v \; = \frac{\pi d^2}{4}\sqrt{2gh}\,,$$

so it depends on the height of the water level above the opening.

How does the volume of water in the upper half of the hourglass change in time? As a function of the height h, its volume is

$$V(h) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(h\tan\alpha\right)^2 h\,,$$

where we determined the cone radius in terms of h using the apex angle as $r = h\tan\alpha$, with $\alpha = 20\,^\circ$ being half of the apex angle. The rate of change of this volume over time is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{3}\tan^2\alpha\frac{\mathrm{d}h^3}{\mathrm{d}t} = \pi\tan^2\alpha h^2\frac{\mathrm{d}h}{\mathrm{d}t}\,.$$

This quantity is directly comparable to the volumetric flow rate Q. Comparing these quantities gives us

$$Q = -\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi d^2}{4}\sqrt{2gh} = -\pi\tan^2\alpha h^2\frac{\mathrm{d}h}{\mathrm{d}t}\,,$$

where the negative sign indicates that the volume of water in the upper part is decreasing. We thus have a differential equation, which we solve by separation of variables:

$$\frac{\pi d^2}{4}\sqrt{2g}\,\mathrm{d}t = -\pi\tan^2\alpha h^{\frac{3}{2}}\,\mathrm{d}h\,,$$

$$\frac{\mathrm{d}^2\sqrt{2g}}{4\tan^2\alpha}t + C = -\frac{2}{5}h^{\frac{5}{2}}\,,$$

where the constant C is determinable from the initial condition that at time $t = 0$, the height in the cone is at its maximum, which we denote as H. Thus, we find $C = -\frac{2}{5}H^{\frac{5}{2}}$.

In the final part, we consider the point at which no water remains in the cone, i.e., $h = 0$. It occurs at $\tau = 2\,\mathrm{min}$, as given in the problem. Substituting the values for t and h into our equation, we obtain

$$H = \left(\frac{5\,\mathrm{d}^2\sqrt{2g}}{8\tan^2\alpha}\tau\right)^{\frac{2}{5}} \doteq 16\,\mathrm{cm}\,.$$

However, we seek the required volume of one chamber, which is then simply

$$V_\mathrm{k} = \frac{\pi}{3} \left(H \tan\alpha\right)^2 H \doteq 550\,\mathrm{ml}\,.$$

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Problem 40 ... rectified efficiency 6 points

We have a Graetz (diode) bridge assembled with diodes. These diodes do not conduct any current until the voltage reaches $U_\mathrm{min} = 0.62\,\mathrm{V}$, after which the current drops (and therefore the diode's resistance) becomes negligible. What is the maximum achievable efficiency (in %) for a device connected to the rectified alternating current from this bridge, given that we use a harmonic source with a maximum voltage of $U_\mathrm{max} = 1.5\,\mathrm{V}$? We want to compare this efficiency to the maximum efficiency of a circuit without any bridge. *Karel was thinking about AC.*

The appliance will achieve maximum power when its power factor is $\cos\varphi = 1$, meaning it behaves like an ideal resistor. Maximum efficiency will thus be the ratio of the power consumed by the appliance to the total power delivered to the bridge. We also assume that the appliance's output power is equal to the power it consumed (which could be true for something like a heater).

Since we are dealing with a rectifier bridge, the appliance's power period is half the source's period, which we denote as T. We only need to consider the time interval from $t_0 = 0\,\mathrm{s}$ to $t_3 = T/2$.

We can express the average power of the source as

$$\begin{aligned}\bar{w} &= \frac{\int_{t_0}^{t_3} U_\mathrm{max} I_\mathrm{max} \sin^2 \frac{2\pi t}{T}\,\mathrm{d}t}{\frac{T}{2}} = U_\mathrm{max} I_\mathrm{max} \frac{1}{\pi} \int_0^\pi \sin^2 x\,dx \\ &= U_\mathrm{max} I_\mathrm{max} \frac{1}{\pi} \left[\frac{1}{2}\left(x - \sin x \cos x\right)\right]_{x=0}^{\pi} = \frac{U_\mathrm{max} I_\mathrm{max}}{2}\,,\end{aligned}$$

where we used the substitution $x = 2\pi t/T$, $\mathrm{d}t = T/(2\pi)\,\mathrm{d}x$.

At times when the voltage is lower than U_min, the power is zero; otherwise, it follows the previous case. This yields integration limits t_1 and t_2:

$$2\pi \frac{t_1}{T} = \arcsin \frac{U_\mathrm{min}}{U_\mathrm{max}} \quad \Rightarrow \quad t_1 = \frac{T}{2\pi} \arcsin \frac{U_\mathrm{min}}{U_\mathrm{max}} \doteq 0.067\,8T\,,$$

$$t_2 = \frac{T}{2} - t_1 = \frac{T}{2}\left(1 - \frac{1}{\pi} \arcsin \frac{U_\mathrm{min}}{U_\mathrm{max}}\right) \doteq 0.432\,T\,.$$

Moreover, we can calculate the average power consumed by the appliance as

$$\bar{w}_\mathrm{G} = \frac{\int_{t_1}^{t_2} U_\mathrm{max} I_\mathrm{max} \sin^2 \frac{2\pi t}{T}\,\mathrm{d}t}{\frac{T}{2}}\,.$$

The maximum efficiency is then

$$\eta = \frac{\bar{w}_\mathrm{G}}{\bar{w}} = \frac{\int_{t_1}^{t_2} U_\mathrm{max} I_\mathrm{max} \sin^2 \frac{2\pi t}{T}\,\mathrm{d}t}{\frac{T}{2} \frac{U_\mathrm{max} I_\mathrm{max}}{2}}\,.$$

We can factor out the maximum voltage and current, so the result depends only on the integration over time or the substituted variable x:

$$\eta = \frac{4}{T}\int_{t_1}^{t_2} \sin^2 \frac{2\pi t}{T}\,\mathrm{d}t = \frac{2}{\pi}\int_{\arcsin \frac{U_{\min}}{U_{\max}}}^{\pi-\arcsin \frac{U_{\min}}{U_{\max}}} \sin^2 x\,\mathrm{d}x \doteq 0.968 = 96.8\,\%\,.$$

Thus, the maximum efficiency with the bridge is approximately 96.8 % of the theoretical maximum efficiency.

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Problem 41 ... all the same 6 points

Imagine a planet where a sidereal day is as long as it is on Earth. Furthermore, on the latitude where Prague is located on Earth, the gravitational acceleration is also the same. However, the radius of the planet is ten times larger than the Earth's radius. What is its average density? Consider both planets to be perfectly homogeneous spheres.

Jarda was in a wrong classroom, but he didn't realize.

Weight is the vector sum of the gravitational and centrifugal forces. The magnitude of the gravitational force is

$$F_\mathrm{G} = \frac{MmG}{R^2} = \frac{4\pi R\rho}{3}Gm\,,$$

where M is the mass of the planet, m is the mass of the person, G is the gravitational constant, R is the radius of the planet, and ρ is its average density. The direction of action is towards the center of the planet.

The magnitude of the centrifugal force at the 50th parallel (where Prague is located) is

$$F_\mathrm{o} = m\omega^2 r = m\left(\frac{2\pi}{T}\right)^2 R\cos 50^\circ\,,$$

where ω is the angular velocity of the rotation, $T = 86\,164\,\mathrm{s}$ is the length of a sidereal day, and r is the distance from the axis of rotation for a point on the 50th parallel.

The magnitude of gravitational acceleration can be expressed in both cases using the law of cosines as

$$g = \frac{1}{m}\sqrt{F_\mathrm{G}^2 + F_\mathrm{o}^2 - 2F_\mathrm{G}F_\mathrm{o}\cos 50^\circ}\,.$$

We will use the subscript $\oplus$ to denote Earth's parameters and the subscript p for the planet's parameters, where they differ (e.g., radius and density). From the equality of gravitational acceleration magnitudes, we obtain the equation

$$F_{\mathrm{G}\oplus}^2 + F_{\mathrm{o}\oplus}^2 - 2F_{\mathrm{G}\oplus}F_{\oplus}\cos 50^\circ = F_{\mathrm{G\ p}}^2 + F_{\mathrm{o\ p}}^2 - 2F_{\mathrm{G\ p}}F_{\mathrm{o\ p}}\cos 50^\circ\,.$$

By substituting the forces and performing several transformations, we obtain

$$\left(\frac{R_\oplus}{R_\mathrm{p}}\right)^2\left(\left(\frac{4\pi\rho_\oplus}{3}G\right)^2 + \cos^2 50^\circ\left(\left(\frac{2\pi}{T}\right)^4 - \frac{32\pi^3\rho_\oplus}{3T^2}G\right)\right) =$$
$$= \left(\frac{4\pi\rho_\mathrm{p}}{3}G\right)^2 + \left(\left(\frac{2\pi}{T}\right)^2\cos 50^\circ\right)^2 - \frac{32\pi^3\rho_\mathrm{p}}{3T^2}G\cos^2 50^\circ\,.$$

The left side of the equation is known, so we denote it as L. For numerical evaluation, this constant can be expressed using data from the constant list as

$$L = \left(\frac{R_\oplus}{R_\mathrm{p}}\right)^2 \left(\left(\frac{M_\oplus}{R_\oplus^3}G\right)^2 + \cos^2 50\,^\circ \left(\left(\frac{2\pi}{T}\right)^4 - \frac{8\pi^2 M_\oplus}{R_\oplus^3 T^2}G\right)\right) = 2.354\,80 \cdot 10^{-14}\,\mathrm{s}^{-1}\,.$$

We are solving only for ρ_p in the previous equation, which appears squared on the right-hand side. Rearranging the equation, we get

$$\left(\frac{4\pi}{3}G\right)^2 \rho_\mathrm{p}^2 - \frac{32\pi^3 G\cos^2 50\,^\circ}{3T^2}\rho_\mathrm{p} - \left(L - \left(\left(\frac{2\pi}{T}\right)^2 \cos 50\,^\circ\right)^2\right) = 0\,,$$

from which the quadratic equation solution gives

$$\rho_\mathrm{p} = 3\frac{4\pi^2\cos^2 50\,^\circ + \sqrt{16\pi^4\cos^4 50\,^\circ + T^4\left(L - \left(\left(\frac{2\pi}{T}\right)^2\cos 50\,^\circ\right)^2\right)}}{4G\pi T^2} = 557\,\mathrm{kg{\cdot}m^{-3}}\,,$$

where we have chosen the + sign in the formula to ensure the density is a positive number.

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Problem 42 ... suspended 6 points

Consider two suspension points at the same height at a distance $d = 50.0\,\mathrm{cm}$ apart. On the first one, we hang a weight of mass $m_1 = 1.30\,\mathrm{kg}$ from a suspension of length $l_1 = 60.0\,\mathrm{cm}$. On the other, we hang a weight of mass $m_2 = 2.10\,\mathrm{kg}$ from a suspension of length $l_2 = 45.0\,\mathrm{cm}$. We then connect both weights with a rope of length $l = 30.0\,\mathrm{cm}$. At what distance below the suspension plane will the lighter weight be located? *Dodo was hanging the laundry.*

The weights will stabilize at an equilibrium position that minimizes the sum of their potential energies. This occurs when their center of gravity is at the lowest possible position. Any deviation of any of the weights from the vertical plane given by the attachment points evidently increases this energy. Thus, we are solving a 2D problem where we minimize the height of the point between the weights at a distance of $lm_2/(m_1 + m_2)$ from the first weight. This problem may be solved mathematically, but it leads to a complex expression that must be solved numerically.

Let us begin with the numerical solution right away – we will draw the entire problem in Geogebra. In the figure 3 the points A and C are the suspension points, G and H are the weights and K is the center of gravity. Points G and H must be at a minimum on the taut ropes, that is on circles with appropriate radii. For a given point G representing the position of the lighter mass, we construct a circle of radius l. This intersection with the circle (C, l_2) defines the position of the heavier mass H. However, there can be two such intersections, giving us a second solution (in red).[3] The great advantage of Geogebra is that a graphic constructed this way is movable. When moving the point G and with the center of gravity path turned on,

[3] The blue color shows the positions of the center of gravity with H to the right of the suspension C, which are clearly not configurations corresponding to the minimal energy, as $l < d$.

we notice that the set of all possible centers of gravity is complex in shape. We are interested in the construction with the center of gravity as low as possible – at a distance of $t = 49.6\,\mathrm{cm}$ from the plane of the suspension, at which the point G is at depth $h_1 = 57.7\,\mathrm{cm}$ below the suspension.

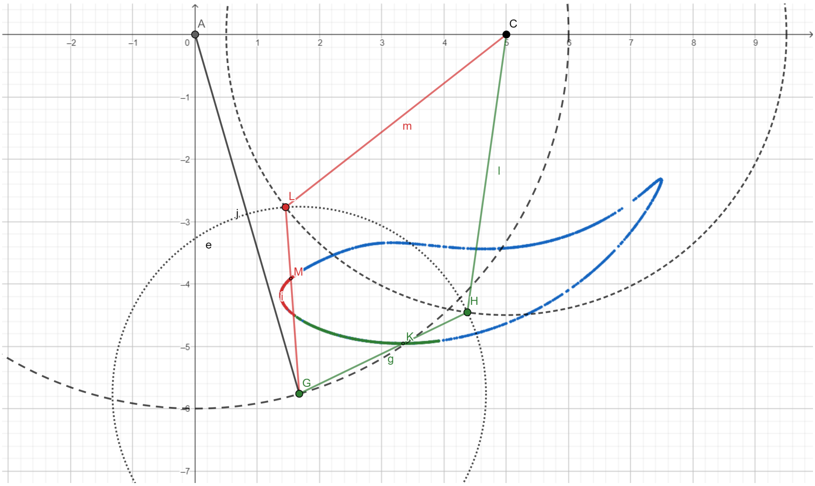

Figure 3: Problem redrawn in Geogebra

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Problem 43 ... border hill with a lookout tower — 7 points

Lego is at a lookout tower between the Czech Republic and Slovakia. He hangs a pulley on the tower, over which he throws a rope, with carts at the ends having masses $m_\mathrm{C} = 24\,\mathrm{kg}$ (placed on the Czech side) and $m_\mathrm{S} = 15\,\mathrm{kg}$ (on the Slovak side). The hill has a slope of $\alpha_\mathrm{S} = 10\,^\circ$ on the Slovak side and $\alpha_\mathrm{C} = 14\,^\circ$ on the Czech side. The rope forms angles $\beta_\mathrm{S} = 15\,^\circ$ and $\beta_\mathrm{C} = 11\,^\circ$ with the hill. What is the acceleration of the cart on the Slovak side, assuming all friction is neglected? If the cart moves upward toward the Czech side, enter a positive value; if downward, a negative one. The pulley and rope are massless, and the rope is perfectly inelastic.

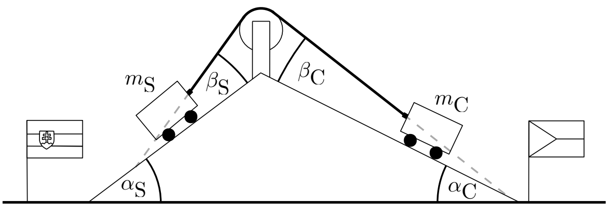

Figure 4: The scheme of the hill.

Lego, of course, imagined the vehicles from FYKOS camp.

In the direction parallel to the slope, the weight of the Slovakian cart has a component $m_\mathrm{S} g \sin\alpha_\mathrm{S}$. If we denote the tension force of the rope as T, its projection in the direction parallel to the slope is $T\cos\beta_\mathrm{S}$. The situation with the Czech trolley is similar. For accelerations in the direction parallel to the slope, denoted as a_S and a_C (both with a positive direction upward), we obtain the following equations of motion:

$$m_\mathrm{S} a_\mathrm{S} = T\cos\beta_\mathrm{S} - m_\mathrm{S} g\sin\alpha_\mathrm{S}$$
$$m_\mathrm{C} a_\mathrm{C} = T\cos\beta_\mathrm{C} - m_\mathrm{C} g\sin\alpha_\mathrm{C}\,.$$

Apart from the unknown accelerations, the equations also contain the tensile force of the rope, which we need to determine. The third equation follows from the rope's inextensibility: the rope cannot stretch or shorten, which means it remains taut. If the trolley on the Slovak side is moved upward by a small amount $\mathrm{d}x_\mathrm{S}$ along the slope, the length of the rope on the Slovak side decreases. What is the decrease in length? We can use a method similar to diffraction on a grating. With sufficiently small displacement, the direction of the rope does not change significantly, so the length of the rope on this side decreases by the projection of the displacement $\mathrm{d}x_\mathrm{S}$ into the direction of the rope, which is $\mathrm{d}x_\mathrm{S}\cos\beta_\mathrm{S}$. We arrive at the same result using the cosine rule. If we denote the original length of the rope as l_S, the new length will be shorter by:

$$l_\mathrm{S} - \sqrt{l_\mathrm{S}^2 + \mathrm{d}x_\mathrm{S}^2 - 2l_\mathrm{S}\,\mathrm{d}x_\mathrm{S}\cos\beta_\mathrm{S}} \approx l_\mathrm{S} - l_\mathrm{S}\sqrt{1 - 2\frac{\mathrm{d}x_\mathrm{S}}{l_\mathrm{S}}\cos\beta_\mathrm{S}} \approx \mathrm{d}x_\mathrm{S}\cos\beta_\mathrm{S}\,,$$

where we consider only the first-order approximation.

On the Czech side, we get the same result. From the inextensibility condition, it holds that the change in the length of the rope on one side is equal to the change in the length on the other side times minus one, so $\mathrm{d}x_\mathrm{S}\cos\beta_\mathrm{S} = -\,\mathrm{d}x_\mathrm{C}\cos\beta_\mathrm{C}$. After differentiating this expression with respect to time twice, we obtain the equality for accelerations

$$a_\mathrm{S}\cos\beta_\mathrm{S} = -a_\mathrm{C}\cos\beta_\mathrm{C}\,.$$

This is the third equation we needed to solve the system of equations. Subsequently, we multiply the first equation of motion by $\cos\beta_\mathrm{C}/\cos\beta_\mathrm{S}$ and subtract it from the second one, eliminating T:

$$m_\mathrm{C} a_\mathrm{C} - m_\mathrm{S} a_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}} = m_\mathrm{S} g\sin\alpha_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}} - m_\mathrm{C} g\sin\alpha_\mathrm{C}\,.$$

Then, we simply substitute for a_C from the equation derived from the inextensibility of the rope, i.e., $a_\mathrm{C} = -a_\mathrm{S}\frac{\cos\beta_\mathrm{S}}{\cos\beta_\mathrm{C}}$, and solve for a_S:

$$-m_\mathrm{C} a_\mathrm{S}\frac{\cos\beta_\mathrm{S}}{\cos\beta_\mathrm{C}} - m_\mathrm{S} a_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}} = m_\mathrm{S} g\sin\alpha_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}} - m_\mathrm{C} g\sin\alpha_\mathrm{C}$$
$$a_\mathrm{S} = g\frac{m_\mathrm{C}\sin\alpha_\mathrm{C} - m_\mathrm{S}\sin\alpha_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}}}{m_\mathrm{C}\frac{\cos\beta_\mathrm{S}}{\cos\beta_\mathrm{C}} + m_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}}} = 0.80\,\mathrm{m{\cdot}s^{-2}}\,.$$

However, an important detail remains. We supposed that the trolleys would accelerate, but theoretically, the trolleys could accelerate upward. If the trolley on one side is very heavy and the angle β on the other side is high enough, the trolley could be lifted off the ground. How can we determine this? We calculate projections of the weight and tensile force acting on the trolley in a direction vertical to the slope. If this projection points downward, the trolley presses against the ground, and it reacts with normal force, causing the trolley to move as we assumed. On the other hand, if the projection points upward, our assumption was incorrect, and the trolley would accelerate upwards.

Let's calculate backward the tensile force T by substituting back into one of the equations of motion:

$$m_\mathrm{S} g \frac{m_\mathrm{C} \sin\alpha_\mathrm{C} - m_\mathrm{S}\sin\alpha_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}}}{m_\mathrm{C}\frac{\cos\beta_\mathrm{S}}{\cos\beta_\mathrm{C}} + m_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}}} = T\cos\beta_\mathrm{S} - m_\mathrm{S} g\sin\alpha_\mathrm{S}\,,$$

$$m_\mathrm{S} g\left(\frac{m_\mathrm{C} \sin\alpha_\mathrm{C} - m_\mathrm{S}\sin\alpha_\mathrm{S}\frac{\cos\beta_\mathrm{C}}{\cos\beta_\mathrm{S}}}{m_\mathrm{C}\frac{\cos^2\beta_\mathrm{S}}{\cos\beta_\mathrm{C}} + m_\mathrm{S}\cos\beta_\mathrm{C}} + \frac{\sin\alpha_\mathrm{S}}{\cos\beta_\mathrm{S}}\right) = T\,,$$

$$m_\mathrm{S} g \frac{m_\mathrm{C} \sin\alpha_\mathrm{C} + m_\mathrm{C}\sin\alpha_\mathrm{S}\frac{\cos\beta_\mathrm{S}}{\cos\beta_\mathrm{C}}}{m_\mathrm{C}\frac{\cos^2\beta_\mathrm{S}}{\cos\beta_\mathrm{C}} + m_\mathrm{S}\cos\beta_\mathrm{C}} = T\,,$$

$$m_\mathrm{S} m_\mathrm{C} g \frac{\sin\alpha_\mathrm{C}\cos\beta_\mathrm{C} + \sin\alpha_\mathrm{S}\cos\beta_\mathrm{S}}{m_\mathrm{C}\frac{\cos^2\beta_\mathrm{S}}{\cos\beta_\mathrm{C}} + m_\mathrm{S}\cos^2\beta_\mathrm{C}} = T\,.$$

The component of this force in the direction perpendicular to the ground is on the Slovak side $T\sin\beta_\mathrm{S}$ (with the positive direction "upward"). The component of gravity in this direction is $-m_\mathrm{S} g\cos\alpha_\mathrm{S}$. The sum of these forces is:

$$m_\mathrm{S} m_\mathrm{C} g \sin\beta_\mathrm{S}\frac{\sin\alpha_\mathrm{C}\cos\beta_\mathrm{C} + \sin\alpha_\mathrm{S}\cos\beta_\mathrm{S}}{m_\mathrm{C}\cos^2\beta_\mathrm{S} + m_\mathrm{S}\cos^2\beta_\mathrm{C}} - m_\mathrm{S} g\cos\alpha_\mathrm{S} = -135\,\mathrm{N}\,,$$

which is negative, meaning that the trolley is pressing against the ground. The trolley on the Czech side accelerates downhill, indicating that the rope does not lift the trolley off the ground, something that can be confirmed by an analogous calculation. The result is therefore the same as the calculation above, so

$$a_\mathrm{S} = 0.80\,\mathrm{m{\cdot}s^{-2}}\,.$$

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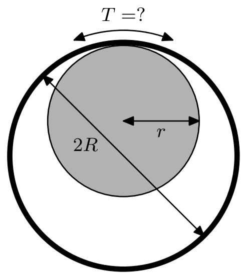
Problem 44 ... loose ring

6 points

A thin ring with radius $r = 5.0\,\mathrm{cm}$ is threaded onto a fixed horizontal rod with a circular cross-section of radius $R = 2.0\,\mathrm{cm}$. Determine the period of the small oscillations after a displacement from the equilibrium position. The ring does not slip relative to the rod.

Jarda is already thinking about a wedding.



The problem may not seem very straightforward at first glance because the point of contact between the larger ring and the smaller one changes during each oscillation. It is therefore not immediately clear how the center of mass of the ring moves and how quickly the ring rotates relative to its center of mass. Let us first describe the motion of the ring.

Consider the point of contact of the two objects, which deviates from the vertical by an angle φ, measured from the center of the rod. The center of the ring must lie on the line connecting this point of contact and the center of the rod, since the two circles are touching. Therefore, the center of the ring is at a distance of $R - r$ from the center of the rod. However, this distance is independent of the angle φ. The center of the ring thus lies on a circle of radius $R - r$ for any φ. To describe the velocity of the ring's center of mass, let us define $\dot{\varphi}$, the time derivative of the angular displacement from equilibrium. Then the velocity of the ring's center of mass is simply

$$v_\mathrm{T} = (R - r)\,\dot{\varphi}\,.$$

Additionally, the ring rotates around its axis. However, it does not rotate with the same angular velocity as its center of mass rotates around the center of the rod. This is because the larger ring rolls on the surface of the smaller one. The point of contact must cover the same distance on the larger ring as on the smaller cylinder. If the point of contact moves on the smaller cylinder by $\Delta\varphi$, this corresponds to $r\Delta\varphi$. For the center of the ring, however, this distance represents only an angle of $\Delta\varphi r/R$, and thus the angle by which the ring rotates is only $\Phi = \varphi - \varphi r/R$. The angular velocity of the ring's rotation is then

$$\dot{\Phi} = \left(1 - \frac{r}{R}\right)\dot{\varphi}\,.$$

We have described the motion of the ring on the cylinder. The period of small oscillations can be found by balancing the kinetic and the potential energy. The kinetic energy is the sum of the translational energy of the cylinder's center of mass and its rotational energy

$$E_\mathrm{k} = \frac{1}{2}mv_\mathrm{T}^2 + \frac{1}{2}J\dot{\Phi}^2 = \frac{1}{2}\left(m\left(R-r\right)^2\dot{\varphi}^2 + mR^2\left(1 - \frac{r}{R}\right)^2\dot{\varphi}^2\right) = \frac{1}{2}2m\left(R-r\right)^2\dot{\varphi}^2\,.$$

The potential energy also depends on the angle, based on the position of the ring's center of mass, as

$$E_\mathrm{p} = mg\left(R-r\right)\left(1-\cos\varphi\right) \approx \frac{1}{2}\varphi^2 mg\left(R-r\right)\,,$$

where we used the small-angle approximation near $\varphi = 0$, where $\cos\varphi \approx 1 - \varphi^2/2$. Thus, we see that the potential energy increases with the square of the coordinate, while the kinetic energy is proportional to the square of the derivative of this coordinate. This situation is entirely analogous to motion in a harmonic oscillator, except that instead of the position x, we use the angle φ, and the proportionality constants are different.

If the energy of a harmonic oscillator is

$$E_{\mathrm{LHO}} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

and the period of oscillations is

$$T_{\mathrm{LHO}} = 2\pi \sqrt{\frac{m}{k}}\,,$$

then in our system, the energy is

$$E = \frac{1}{2} 2m \left(R - r\right)^2 \dot{\varphi}^2 + \frac{1}{2} m g \left(R - r\right) \varphi^2\,,$$

and the period is

$$T = 2\pi \sqrt{\frac{2m \left(R - r\right)^2}{m g \left(R - r\right)}} = 2\pi \sqrt{\frac{2 \left(R - r\right)}{g}} = 0.49\,\mathrm{s}\,.$$

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Problem 45 ... double detection — 7 points

Jindra is measuring the decay of a radioactive isotope using an energy-sensitive detector. He knows that one decay releases the energy E. During the measurement time $T = 300\,\mathrm{s}$ he detects $N_1 = 9\,728\,350$ events with energy E and $N_2 = 166\,545$ events with energy $2E$. The number of events with energy $3E$ or higher is negligible. Particle detectors have a so-called dead time during which they process the signal from the previous particle. Specifically, Jindra's detector also records particles that arrive during the dead time, but it cannot detect them as separate particles, instead, it adds their energy to the first particle. What is the dead time of Jindra's detector? *Jindra is counting decays.*

The dead time during which detectors process the signal from the previous particle can manifest in various ways. For example, the detector may not be able to receive any additional signals at all, and all events occurring during the dead time will be lost. In our example, it is evident that the detector does receive signals even during the dead time, but it adds their energy to the currently processed signal and cannot distinguish them as separate events. N_1 detections with energy E correspond to single hits, and N_2 detections with energy $2E$ correspond to double hits, where a new particle enters the detector during the dead time while the detector is still processing the signal from the previous particle.

The average particle frequency is

$$f = \frac{N_1 + 2N_2}{T}\,.$$

We denote the yet-unknown dead time as τ. Since $N_2 \ll N_1$, the number of events n during the dead time after a hit can be described by the Poisson distribution

$$P(n) = \frac{(f\tau)^n}{n!} \mathrm{e}^{-f\tau}\,.$$

The most likely scenario is $n = 0$, meaning no additional hits occur during the dead time, with a probability of

$$P(0) = \mathrm{e}^{-f\tau} \approx 1 - f\tau,$$

where we used the approximation $f\tau \ll 1 \implies \mathrm{e}^{-f\tau} \approx 1$. The probability that $n = 1$ hits occur during the dead time is

$$P(1) = f\tau \mathrm{e}^{-f\tau} \approx f\tau,$$

where we used the same approximation.

Now let us describe the problem using the language of the probability theory. Each particle initiates an interval of length τ, during which no additional particle arrives with probability $P(0) = 1 - f\tau$, and exactly one additional particle arrives with probability $P(1) = f\tau$. The number of "random trials", i.e., the number of initiated intervals of length τ, during Jindra's measurement was

$$N = N_1 + N_2.$$

The number of double events during the entire measurement of duration T was

$$N_2 = NP(1) = Nf\tau = \frac{(N_1 + N_2)(N_1 + 2N_2)}{T}\tau.$$

The dead time of Jindra's detector is thus

$$\tau = \frac{N_2}{(N_1 + N_2)(N_1 + 2N_2)}T = 5.02 \cdot 10^{-7}\,\mathrm{s} \doteq 500\,\mathrm{ns}.$$

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Problem 46 ... back to the U-tube — 6 points

Let us consider a U-tube with an internal cross-sectional area $0.80\,\mathrm{cm}^2$, into which we pour $110\,\mathrm{ml}$ of water. The length of each of the symmetrically placed arms of the tube is $90\,\mathrm{cm}$. We place an inflated balloon over one end of the tube, causing the water level in the other arm to rise by $\Delta h = 4.0\,\mathrm{cm}$. How much do we need to compress the balloon, i.e., by how much do we need to decrease its volume, so that the water starts overflowing from the other arm? The volume of the balloon just before compression is $V_0 = 750\,\mathrm{ml}$. Neglect the volume of the horizontal part and the curved sections of the U-tube. *Jarda was devising an experiment for FYKOS.*

At the interface between water and air in the arm with the balloon, the pressures must be equal. On one side acts the hydrostatic pressure of the water with height $2\Delta h$, balancing the air pressure from the inflated balloon. This pressure must be $p_h = \rho g 2\Delta h \doteq 783\,\mathrm{Pa}$.

For the water to start flowing out, the water level in the arm without the balloon must rise to a height l, so the height difference between the arms must be

$$\delta H = l - (V/S - l) = 2l - V/S = 42.5\,\mathrm{cm}\,.$$

The required pressure is therefore

$$p_H = \rho g \Delta H = \rho g\left(2l - V/S\right) \doteq 4\,160\,\mathrm{Pa}\,.$$

The change in pressure is caused by compressing the balloon, reducing its volume. The volume of air after placing the balloon in the arm is

$$V_h = V_0 + S\left(l - V/(2S) + \Delta h\right) ,$$

where V_0 is the volume of the balloon. After compression, we denote the volume of the balloon as V_B, so the total air volume is

$$V_H = V_B + 2lS - V .$$

From the ideal gas law, we have $V_h p_h = V_H p_H$, leading to

$$\Delta V = V_B - V_0 = V_h \frac{p_h}{p_H} - 2lS + V - V_0 ,$$

$$\Delta V = \left(V_0 + S\left(l - \frac{V}{2S} + 2\Delta h\right)\right) \frac{2\Delta h}{\left(2l - \frac{V}{S}\right)} - 2lS + V - V_0 = -640\,\mathrm{ml} .$$

Hence, the volume of the balloon needs to be reduced by 640 ml.

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Problem 47 ... acceleration of an idealized car 6 points

Let us consider a car with a power output of $P = 44\,\mathrm{kW}$ and a mass of $m = 1\,400\,\mathrm{kg}$, which is subjected to a total resistive force of $F = 950\,\mathrm{N}$ (we assume this force is constant – as at low speeds, air resistance is negligible). In how much time will the car accelerate from 0 to $v_\mathrm{f} = 18\,\mathrm{km{\cdot}h^{-1}}$? *Lego sometimes ponders like this.*

We can determine the acceleration by dividing the total force by the car's mass. Moreover, we can calculate the car's power as the force it exerts on the road multiplied by its velocity. Given that we know the power, if the car's velocity is v, then the driving force is P/v. To find the total force, we must subtract the opposing drag force F. Thus, we construct the differential equation as

$$\frac{P}{v} - F = m\frac{\mathrm{d}v}{\mathrm{d}t} .$$

We will solve this equation using the method of separation of variables

$$\mathrm{d}t = m\frac{\mathrm{d}v}{\frac{P}{v} - F}$$

$$\int_0^{t_\mathrm{f}} \mathrm{d}t = m\int_0^{v_\mathrm{f}} \frac{\mathrm{d}v}{\frac{P}{v} - F} ,$$

$$t_\mathrm{f} = m\left[-\frac{v}{F} - \frac{P}{F^2}\ln(P - Fv)\right]_0^{v_\mathrm{f}} ,$$

$$t_\mathrm{f} = m\left(\frac{P}{F^2}(\ln P - \ln(P - Fv_\mathrm{f})) - \frac{v_\mathrm{f}}{F}\right) ,$$

$$t_\mathrm{f} = \frac{m}{F}\left(\frac{P}{F}\ln\left(\frac{P}{P - Fv_\mathrm{f}}\right) - v_\mathrm{f}\right) \doteq 0.43\,\mathrm{s} .$$

We observe that time is proportional to the mass of the car. The time diverges for $P = Fv_f$, indicating the limit where the car accelerates to its maximum achievable speed, at which the driving and opposing forces balance out and the net force approaches zero. The subtraction of v_f might seem surprising at first glance, but if we look more closely at the first term, we can calculate that v_f is the first term of its Taylor series expansion and it makes perfect sense that it should drop out since this term does not depend on P at all. As the only term left will be the second term of the Taylor expansion, the resistive force F will exactly cancel out and then for the other terms the force will already be in the numerator.

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Problem 48 ... fishing 7 points

A point particle with a mass of $m = 1.85\,\mathrm{kg}$ lies on a horizontal surface. It is connected to a tight rope that ends at a winch located at a horizontal distance $L = 20.4\,\mathrm{m}$ from the point particle and at a height $H = 6.35\,\mathrm{m}$. The winch starts winding the rope at a speed of $v = 3.25\,\mathrm{m{\cdot}s^{-1}}$. How long will it take for the point particle to lift off from the surface if the coefficient of friction between it and the surface is $f = 0.415$? Jarda was winding up a cable from his mower.

First, we express the position of the mass point as a function of time from the start of the unwinding t

$$x(t) = \sqrt{(l_0 - vt)^2 - H^2}\,,$$

where $l_0 = \sqrt{H^2 + L^2}$ is the initial length of the rope. By differentiating the position, we obtain the velocity of the mass point

$$v_x(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = (-v\,(l_0 - vt))\,\frac{1}{\sqrt{(l_0 - vt)^2 - H^2}}\,,$$

and by differentiating the velocity with respect to time, we find the acceleration $a_x(t)$

$$a_x(t) = \frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = \frac{v^2}{\sqrt{(l_0 - vt)^2 - H^2}}\left(1 - \frac{(l_0 - vt)^2}{\left((l_0 - vt)^2 - H^2\right)}\right)\,.$$

We then find the net force acting on the mass point in the horizontal direction as

$$F_x(t) = ma_x(t) = -T_x(t) + f\,(mg - T_y(t))\ ,$$

where $T_x(t)$ is the horizontal component of the tension in the rope. We can find this using the acceleration and the knowledge of the frictional force. Once we know the horizontal component of the tension, we can calculate the vertical component using the angle between the rope and the horizontal plane

$$T_y(t) = T_x(t)\tan\alpha = \frac{H}{x(t)}T_x(t)\ .$$

We substitute this into the previous equation and obtain

$$F_x(t) = ma_x(t) = -\frac{x(t)}{H}T_y(t) + (mg - T_y(t)))\,f \quad\Rightarrow\quad T_y(t) = \frac{mgf - ma_x(t)}{\frac{x(t)}{H} + f}\,.$$

When $T_y(t) > mg$ is satisfied, the object lifts off the surface. We substitute into the equation and simplify it

$$T_y(t) = mg = H \frac{mgf - m\left(\frac{v^2}{\sqrt{(l_0-vt)^2-H^2}}\left(1-\frac{(l_0-vt)^2}{\left((l_0-vt)^2-H^2\right)}\right)\right)}{\sqrt{\left(l_0-vt\right)^2-H^2}+fH} \,,$$

$$g\left((l_0-vt)^2-H^2\right) = Hv^2\left(\frac{\left(l_0-vt\right)^2}{\left((l_0-vt)^2-H^2\right)}-1\right) \,,$$

$$\frac{g}{Hv^2}\left((l_0-vt)^2-H^2\right)^2 = H^2 \,,$$

$$\left((l_0-vt)^2-H^2\right) = \sqrt{\frac{H^3v^2}{g}} \,,$$

$$\Rightarrow \quad t = \frac{l_0-\sqrt{\sqrt{\frac{H^3v^2}{g}}+H^2}}{v} = 4.25\,\mathrm{s} \,.$$

The mass point lifts off the surface at $t = 4.25\,\mathrm{s}$.

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Problem 49 ... the oppressed air 8 points

What is the minimal volume at temperature $t_\mathrm{v} = 25.0\,^\circ\mathrm{C}$ to which $V_1 = 1.00\,\mathrm{m}^3$ of air at normal pressure p_a and at same temperature can be compressed? We have a water boiler of volume $V = 100\,\mathrm{l}$ and temperature $t_\mathrm{b} = 80.0\,^\circ\mathrm{C}$ as an energy source, and a large lake of water at a temperature $t_\mathrm{j} = 5.00\,^\circ\mathrm{C}$ as a cooler. Assume that the air behaves as a monatomic ideal gas. All hypothetical machines used to compress the gas must return to their original state at the end (it is not necessary to consider the specific design of these auxiliary machines to solve the problem). *Dodo needed compressed air.*

From the laws of thermodynamics, we know that when extracting heat from a substance, there is a limit to the amount of work that can be derived from this heat, if we want the machines used to operate cyclically. Furthermore, it turns out that this work is maximized when all considered processes are reversible.

The first intuition in solving our problem might be to design such a reversible process. Here, we would most likely need to connect a Carnot engine[4] between the boiler and the lake and use the work to compress the air. The compression must again be reversible, which can be achieved, for example, through an isothermal process (an isothermal process is useful because the air is supposed to have the same temperature at the end). During isothermal compression, heat must be extracted from the air. This heat can again be used to perform work! Therefore, we would need to use the same reasoning: to ensure that all processes are reversible, we would have to attach another Carnot engine to the air, further compress it, and transfer the waste heat to the lake.

[4] A Carnot engine is an example of a reversible engine that extracts heat Q_1 from one substance at temperature T_1, performs work W, and releases waste heat Q_2 at temperature T_2. One of the important results of thermodynamics is that the efficiency of such a reversible process is $W/Q_1 = 1 - T_1/T_2$.

In such a construction, we know the efficiencies of the Carnot engines and the laws for an ideal gas, so we should be able to calculate the resulting pressure. However, this construction is very complex, and the procedure would be painstaking. Fortunately, there is a better method called the *maximum work theorem*, which leverages the fact that the total entropy remains unchanged in reversible processes.

When using the maximum work theorem, we do not need to focus on the course of the entire process, but only on the initial and final states of the entire system (in this case, the boiler, lake, and air). We know that the maximum work (i.e., the maximum compression of the gas) is obtained if the total entropy does not change. Additionally, we still have the general validity of the law of conservation of energy.

Let us now gradually express the respective changes in energy and entropy for the individual parts of our system. We start with the lake. When the waste heat Q is released, the volume of the water does not change, so no work is done. The change in internal energy is therefore $\Delta U_\mathrm{j} = Q$. Additionally, the lake has a very large heat capacity. Its temperature thus does not change, and from the relation for entropy change,

$$\mathrm{d}S = \frac{\delta Q}{T}$$

we have that the entropy change of the lake is $\Delta S_\mathrm{j} = Q/T_\mathrm{j}$, where T_j is the temperature of the lake in Kelvin, i.e., $T_\mathrm{j} \doteq 278\,\mathrm{K}$ (the letter T with the corresponding index will henceforth always denote the thermodynamic temperature of the given system).

Let us look at the boiler. It starts at temperature T_b and ends at the temperature of the lake T_j. Again, no work is done during heat extraction, but the water has a finite heat capacity $C = cm = c\rho V \doteq 417.6\,\mathrm{kJ{\cdot}K^{-1}}$. The internal energy changes by $\Delta U_\mathrm{b} = C(T_\mathrm{b} - T_\mathrm{j})$. For the change in entropy, we have

$$\mathrm{d}S = \frac{\delta Q}{T} = C\frac{\mathrm{d}T}{T}\,,$$

which, upon integration from the initial to the final temperature, gives

$$\Delta S_\mathrm{b} = C \cdot \ln\frac{T_\mathrm{j}}{T_\mathrm{b}}\,.$$

The air remains. We know that the final temperature must be the same as the initial one, so the internal energy does not change (we assume the air is an ideal gas and that internal energy depends only on temperature). The equation for the conservation of energy is

$$Q - C(T_\mathrm{b} - T_\mathrm{j}) = 0\,.$$

We now have everything needed to numerically calculate the change in the entropy of the air because, from the entropy conservation equation,

$$\frac{Q}{T_\mathrm{j}} - C\ln\frac{T_\mathrm{b}}{T_\mathrm{j}} + \Delta S_\mathrm{v} = 0$$

we easily express

$$\Delta S_\mathrm{v} = -C\left(\frac{T_\mathrm{b} - T_\mathrm{j}}{T_\mathrm{j}} - \ln\frac{T_\mathrm{b}}{T_\mathrm{j}}\right) = -12.9\,\mathrm{kJ{\cdot}K^{-1}}$$

The entropy change is therefore negative, which corresponds precisely to the expectation that we compressed the gas without changing the temperature.

What remains is to express the change in entropy of the ideal gas as a function of the volume change. Here, we can find the expression for the entropy of a single-component ideal gas online:

$$S = Ns_0 + Nk_\mathrm{B} \ln \left[\left(\frac{U}{U_0}\right)^c \left(\frac{V}{V_0}\right) \left(\frac{N_0}{N}\right)^{c+1} \right],$$

where s_0 is a constant such that $N_0 s_0$ is the entropy of a reference ideal gas with energy U_0, volume V_0, and number of particles N_0 (in this area of thermodynamics, only entropy changes are important, so we do not need to think particularly about these reference values).

From the expression for entropy, it is easy to see that if only V changes, the resulting change of entropy will be

$$\Delta S_\mathrm{v} = Nk_\mathrm{B} \ln \frac{V_2}{V_1} = \frac{p_\mathrm{a} V_1}{T_\mathrm{v}} \ln \frac{V_2}{V_1}\,,$$

from which we simply calculate

$$V_2 = V_1 \exp\left(\frac{T_\mathrm{v} \Delta S_\mathrm{v}}{p_\mathrm{a} V_1}\right) = 3.22 \cdot 10^{-17}\,\mathrm{m}^3 = 3.22 \cdot 10^{-8}\,\mathrm{mm}^3\,,$$

which corresponds to a compression into a container of approximately $1\,\mathrm{\mu m}$ dimensions.

How should we proceed if the formula for the entropy of an ideal gas were not available? We could partially derive it. We know that the state of the gas (and thus its entropy) is determined by three (suitable) state variables. In our case, we know the temperature and number of particles at the end of the process, so that suffices to express the change in entropy as a function of the volume change.

Consider an isothermal process. It is suitable because neither the temperature nor the number of particles change, which is precisely what we want. During an isothermal process, some heat Q_1, equal to the work done, is extracted from the gas. For the work in an isothermal process, we have

$$W = -\int_{V_1}^{V_2} p\,\mathrm{d}V \; = p_\mathrm{a} V_1 \ln \frac{V_1}{V_2}\,,$$

which we substitute into the entropy change $\Delta S_\mathrm{v} = -Q_1/T_\mathrm{v}$, yielding the same result. Note that the isothermal process in this procedure is merely a tool for deriving the relationship for the entropy change. In reality, we are not claiming that compression in our problem must occur isothermally. This is better seen from the procedure that directly uses the entropy of the ideal gas.

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Problem 50 ... optimally excited quantum oscillator 7 points

Lego created a one-dimensional quantum linear oscillator with $\omega = 1.17 \cdot 10^{13}\,\mathrm{s}^{-1}$. He connects it to a thermal reservoir of temperature T and waits until equilibrium is reached. What should this temperature be to maximize the probability of the electron occupating the first excited state? *According to Lego, the name of this problem is very photosynthetic.*

When equilibrium is reached, the probabilities of the states will be given by the Boltzmann distribution

$$p_n = \frac{1}{Z}\mathrm{e}^{-\frac{E_n}{k_\mathrm{B}T}}\,,$$

where k_B is the Boltzmann constant and T is the temperature of the thermal reservoir with which the system is in equilibrium. The partition function Z is, among other things, a normalizing constant whose magnitude can be obtained from the condition $\sum_{n=0}^{\infty} p_n = 1$

$$Z = \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{E_n}{k_\mathrm{B}T}} = \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{\hbar\omega(n+1/2)}{k_\mathrm{B}T}} = \mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B}T}} \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}n} = \frac{\mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B}T}}}{1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}}\,,$$

where in the last step we just sum up the geometric series. Thus, the probability of the electron occupating the first excited state is

$$p_1 = \frac{1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}}{\mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B}T}}}\mathrm{e}^{-\frac{\hbar\omega(1+1/2)}{k_\mathrm{B}T}} = \left(1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right)\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\,,$$

and our task is to find out for which T this value will be maximal.

We can straightforwardly differentiate with respect to T, set the result equal to 0, and get T for which p_1 is maximal from the obtained equation. However, for the lazier among us who don't feel like using the chain rule, here's a trick: substitute $x = \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}$, then $p_1 = (1-x)x$. This is the parabola that has a maximum value for $x = 1/2$. Substituting the substitution back in gives us the equation for T

$$\frac{1}{2} = \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}} \qquad \Rightarrow \qquad T = \frac{\hbar\omega}{k_\mathrm{B}\ln 2} = 129\,\mathrm{K}\,,$$

for this temperature, the first excited state will be occupied with the highest probability.

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Problem X.1 ... floating solar sail 4 points

What would the surface density of a solar sail need to be for the pressure of the solar radiation to exactly balance the gravitational force of the Sun? The surface of a solar sail is perfectly reflective, and the plane of the sail is perpendicular to the Sun.

Jindra was interested, what area he would need to expand to freely float in the Solar system.

An incoming photon with momentum $p_\mathrm{before} = E/c$ strikes the sail and reflects, changing its momentum to $p_\mathrm{after} = -E/c$. The total momentum must be conserved, so the sail's momentum changes by $\Delta p_\mathrm{sail} = p_\mathrm{before} - p_\mathrm{after} = 2E/c$. If we consider the total change in momentum of all incoming photons per unit of time, we obtain the force

$$F_\mathrm{rad} = \frac{2IS}{c}\,,$$

where I is the irradiance (power consumption per unit of area), S is the area of the sail and c is the speed of light. The irradiance at a distance r from the center of the Sun is

$$I = \frac{L}{4\pi r^2}\,,$$

where $L = 3.83 \cdot 10^{26}\,\mathrm{W}$ is the Sun's luminosity. We observe that the radiation force is proportional to $1/r^2$, just like the gravitational force. The two forces come to a balance when

$$F_{\mathrm{rad}} = \frac{LS}{2\pi c r^2} = \frac{GMm}{r^2},$$

where G is the gravitational constant, $M = 1.99 \cdot 10^{30}\,\mathrm{kg}$ is the Sun's mass, and m is the total mass of the solar sail. The critical area density of the solar sail is

$$\frac{m}{S} = \frac{L}{2\pi c M G} = 1.531 \cdot 10^{-3}\,\mathrm{kg{\cdot}m^{-2}} \doteq 1.53\,\mathrm{g{\cdot}m^{-2}}.$$

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Problem X.2 ... solar sail on the way to the stars — 5 points

Consider a solar sail of mass $m = 10\,\mathrm{kg}$ and a sail of area $S = 10\,000\,\mathrm{m^2}$. The sail begins its journey in Earth's orbit with zero initial velocity relative to the Sun. What will be the magnitude of the speed of the solar sail at infinity? The surface of a solar sail is perfectly reflective, and the plane of the sail is always perpendicular to the Sun.

Jindra wanted to expand so much, that he could fly to the stars.

The solar sail is subjected to radiation pressure acting away from the Sun and the gravitational force pulling towards the Sun. The radiation force acting on a perfectly reflective sail of area S, illuminated perpendicularly, is

$$F = \frac{2IS}{c} = \frac{LS}{2\pi c r^2}\,,$$

where $L = 3.83 \cdot 10^{26}\,\mathrm{W}$ is the luminosity of the Sun, c is the speed of light, and r is the distance from the center of the Sun. The resulting force acting on the sail is given as the difference between the gravitational force (directed towards the Sun) and the radiation force (directed away from the Sun)

$$F(r) = \frac{LS}{2\pi c r^2} - \frac{GMm}{r^2}\,,$$

where G is the gravitational constant, $M = 1.99 \cdot 10^{30}\,\mathrm{kg}$ is the mass of the Sun, and m is the mass of the sail. The radial attractive force $F(r) = -km/r^2$ is thus modified compared to classical celestial mechanics by the constant

$$k = GM - \frac{LS}{2\pi c m} = -7.051 \cdot 10^{19}\,\mathrm{m^3{\cdot}s^{-2}}\,.$$

After substituting the given values, we can confirm that $k < 0$, meaning the force is repulsive.

The potential energy in this field with respect to the distance r from the Sun is given as

$$E_{\mathrm{p}} = -\frac{km}{r}\,.$$

The probe's initial kinetic energy is zero. Its initial potential energy is

$$E_{\mathrm{p},0} = -\frac{km}{a_{\mathrm{Z}}}\,,$$

where $a_\mathrm{Z} = 1.496 \cdot 10^{11}\,\mathrm{m}$ is the radius of Earth's orbit. At infinity, the potential energy of the probe is zero, so all initial potential energy will convert into kinetic energy, leading to

$$\frac{1}{2}mv^2 = -\frac{km}{a_\mathrm{Z}}\,,$$

$$v = \sqrt{-\frac{2k}{a_\mathrm{Z}}} = 3.070 \cdot 10^4\,\mathrm{m{\cdot}s^{-1}} \doteq 30.7\,\mathrm{km{\cdot}s^{-1}}\,.$$

Thus, the magnitude of the speed of the solar sail at infinity is $v = 30.7\,\mathrm{km{\cdot}s^{-1}}$.

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Problem X.3 ... solar sail on the way to the stars II — 6 points

Consider a solar sail of a mass of $m = 10\,\mathrm{kg}$ and a sail of area $S = 10\,000\,\mathrm{m}^2$. Initially, the solar sail orbits the Sun along Earth's orbit, but far enough to be outside Earth's gravitational influence. Suddenly, it unfurls its sail very quickly. What will the speed of the solar sail be at infinity? The surface of the solar sail is perfectly reflective, and the plane of the sail is always oriented perpendicular to the Sun. Assume Earth's orbit is circular.

Jindra felt that the Earth was attracting him too strongly.

The solar sail is subjected to radiation pressure acting away from the Sun and the gravitational force pulling towards the Sun. The total force is

$$F(r) = \frac{LS}{2\pi c r^2} - \frac{GMm}{r^2}\,,$$

where $L = 3.83 \cdot 10^{26}\,\mathrm{W}$ is the luminosity of the Sun, S is the area of the solar sail, c is the speed of light, G is the gravitational constant, $M = 1.99 \cdot 10^{30}\,\mathrm{kg}$ is the mass of the Sun, m is the mass of the sail and r is the distance from the center of the Sun. This relation is derived in detail in the problem titled "solar sail on the way to the stars".

The radial attractive force $F(r) = -km/r^2$ is modified compared to classical celestial mechanics by the constant

$$k = GM - \frac{LS}{2\pi c m} = -7.051 \cdot 10^{19}\,\mathrm{m^3{\cdot}s^{-2}}\,.$$

By substituting the given values, we verified that $k < 0$, meaning the force is repulsive.

The potential energy in this field, depending on the distance r from the Sun, is

$$E_\mathrm{p} = -\frac{km}{r}.$$

Initially, the probe was moving along a circular orbit with a radius $a_\mathrm{Z} = 1.496 \cdot 10^{11}\,\mathrm{m}$, so its initial velocity is

$$v_0 = \sqrt{\frac{GM}{a_\mathrm{Z}}}.$$

The initial kinetic energy of the probe is

$$E_{\mathrm{k},0} = \frac{1}{2}mv_0^2 = \frac{1}{2}m\frac{GM}{a_\mathrm{Z}}.$$

The initial velocity vector is perpendicular to the direction towards the Sun. The initial potential energy is

$$E_{\mathrm{p},0} = -\frac{km}{a_\mathrm{Z}}.$$

At infinity, the potential energy of the probe will be zero, therefore all the initial potential energy will convert into kinetic energy. Due to the conservation of angular momentum, the perpendicular component of the velocity will be zero at infinity, and the velocity vector will be radial

$$\frac{1}{2}mv^2 = -\frac{km}{a_\mathrm{Z}} + \frac{1}{2}m\frac{GM}{a_\mathrm{Z}}\,,$$

$$v = \sqrt{-\frac{2k}{a_\mathrm{Z}} + \frac{GM}{a_\mathrm{Z}}} = 4.278 \cdot 10^4\,\mathrm{m{\cdot}s^{-1}} \doteq 42.8\,\mathrm{km{\cdot}s^{-1}}\,.$$

Thus, the velocity of the solar sail at infinity will be $v = 42.8\,\mathrm{km{\cdot}s^{-1}}$.

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Problem X.4 ... solar sail on the way to Mars — 7 points

What is the necessary surface density of a solar sail, in order for it to travel from Earth's orbit to Mars' orbit via a Hohmann transfer orbit? The solar sail initially orbits the Sun in the same orbit as Earth, then deploys its sail. The surface of the solar sail is perfectly reflective, and the sail plane is always oriented perpendicular to the Sun. The solar sail has no rocket engines and relies solely on its reflectivity. Ignore Earth's gravitational influence at the start of the solar sail's journey, and assume the orbits of both Earth and Mars around the Sun are circular. *Jindra wanted to surprise Teri with a problem about the Hohmann transfer orbit.*

The Hohmann trajectory between two circular orbits is a half-ellipse with a periapsis at the inner orbit and an apoapsis at the outer orbit. The Hohmann trajectory is fuel-efficient and is used for missions to nearby planets such as Venus and Mars. For more distant planets in the solar system, such as Jupiter or Saturn, gravitational maneuvers are used.

The solar sail is subjected to radiation pressure acting away from the Sun and gravitational force pulling towards the Sun. The total force is

$$F(r) = \frac{LS}{2\pi c r^2} - \frac{GMm}{r^2},$$

where $L = 3.83 \cdot 10^{26}\,\mathrm{W}$ is the luminosity of the Sun, S is the area of the solar sail, c is the speed of light, G is the gravitational constant, $M = 1.99 \cdot 10^{30}\,\mathrm{kg}$ is the mass of the Sun, m is the mass of the sail, and r is the distance from the center of the Sun. This relation is derived in detail in the problem titled "solar sail on the way to the stars".

The radial attractive force $F(r) = -km/r^2$ is modified compared to classical celestial mechanics by the constant

$$k = GM - \frac{LS}{2\pi cm} = -7.051 \cdot 10^{19}\,\mathrm{m^3{\cdot}s^{-2}}\,.$$

However, this new force is still directly proportional to $1/r^2$, just like the classical gravitational force, so the sail will move along a conic section, just as in ordinary celestial mechanics. Only the constant k is smaller.

The potential energy of the sail at a distance r from the Sun is

$$E_\mathrm{p} = -\frac{km}{r}.$$

The total mechanical energy on the elliptical orbit is analogous to the gravitational field

$$E = -\frac{km}{2a},$$

where a is the semi-major axis of the ellipse. For the Hohmann trajectory between Earth's and Mars's orbits, the distance at periapsis is $r_\mathrm{p} = a_\mathrm{Z} = 1.496 \cdot 10^{11}\,\mathrm{m}$, the distance at apoapsis is $r_\mathrm{a} = a_\mathrm{M} = 2.279 \cdot 10^{11}\,\mathrm{m}$ and the semi-major axis is $a = (a_\mathrm{Z} + a_\mathrm{M})/2$. The velocity at the periapsis is the same as Earth's orbital velocity

$$v_\mathrm{p} = \sqrt{\frac{GM}{a_\mathrm{Z}}}.$$

However, once the probe unfolds the sail, the constant of the attractive force k changes, and the probe's orbit shifts from circular to elliptical.

For the probe to reach Mars, its parameters must satisfy the following equation, which we will manipulate

$$\begin{aligned}
-\frac{km}{a_\mathrm{Z} + a_\mathrm{M}} &= -\frac{km}{a_\mathrm{Z}} + \frac{1}{2}mv_\mathrm{p}^2,\\
-\frac{k}{a_\mathrm{Z} + a_\mathrm{M}} &= -\frac{k}{a_\mathrm{Z}} + \frac{GM}{2a_\mathrm{Z}},\\
k\left(\frac{1}{a_\mathrm{Z}} - \frac{1}{a_\mathrm{Z} + a_\mathrm{M}}\right) &= \frac{GM}{2a_\mathrm{Z}},\\
k\frac{a_\mathrm{M}}{a_\mathrm{Z}(a_\mathrm{Z} + a_\mathrm{M})} &= \frac{GM}{2a_\mathrm{Z}},\\
k = GM - \frac{LS}{2\pi cm} &= \frac{GM(a_\mathrm{Z} + a_\mathrm{M})}{2a_\mathrm{M}},\\
\frac{L}{2\pi GcM}\frac{S}{m} &= 1 - \frac{a_\mathrm{Z} + a_\mathrm{M}}{2a_\mathrm{M}},\\
\frac{m}{S} &= \frac{L}{2\pi GcM}\left(1 - \frac{a_\mathrm{Z} + a_\mathrm{M}}{2a_\mathrm{M}}\right)^{-1},\\
\frac{m}{S} &= 8.912 \cdot 10^{-3}\,\mathrm{kg{\cdot}m^{-2}} \doteq 8.91\,\mathrm{g{\cdot}m^{-2}}.
\end{aligned}$$

Thus, the surface density of the solar sail would need to be $8.91\,\mathrm{g{\cdot}m^{-2}}$, to travel from Earth to Mars along the Hohmann trajectory.

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Problem H.1 ... air under water – simplified to the maximum 4 points

We have a cube of side length $a = 10\,\mathrm{cm}$ and density $\rho_\mathrm{k} = 150\,\mathrm{kg{\cdot}m^{-3}}$. Starting from a position where the bottom face of the cube is just touching the water surface, we submerge the cube in the water by pushing it straight down (without any rotation) by $h = 6.0\,\mathrm{cm}$. How much work is performed in this process? Assume that the surface area S of the water is enormous ($S \gg a^2$), so the water level does not change during the process. *Lego was correcting...*

Using energy considerations

The cube has a mass $m_\mathrm{k} = a^3\rho_\mathrm{k}$, so when it is submerged by $\Delta h_\mathrm{k} = -h$, its potential energy decreases by

$$\Delta E_\mathrm{pk} = m_\mathrm{k} g \Delta h_\mathrm{k} = -a^3\rho_\mathrm{k} g h\,.$$

We have thus displaced water of volume a^2h, and of mass $m_\mathrm{w} = a^2h\rho_\mathrm{w}$. The centre of gravity of this block of water was $h/2$ below the surface before the immersion. Since we assume that the water reservoir is large and thus the water level does not change when the cube is submerged, this water was displaced exactly to the level of the surface, so the height of its center of gravity increased by $\Delta h_\mathrm{w} = h/2$. Thus the potential energy of the water has increased by

$$\Delta E_\mathrm{pw} = m_\mathrm{w} g \Delta h_\mathrm{w} = a^2 h \rho_\mathrm{w} g \frac{h}{2}\,.$$

In total, the potential energy therefore increases by

$$\Delta E_\mathrm{p} = \Delta E_\mathrm{pw} - \Delta E_\mathrm{pk} = a^2 h\rho_\mathrm{w} g\frac{h}{2} - a^3\rho_\mathrm{k} g h = g a^2 h\left(\rho_\mathrm{w}\frac{h}{2} - \rho_\mathrm{k} a\right) = 0.088\,\mathrm{J}\,.$$

Using forces

The cube is continuously affected by the gravitational force $F_\mathrm{g} = m_\mathrm{k} g = a^3\rho_\mathrm{k} g$. At the same time, a buoyant force will be acting on it during the submersion. When the cube is submerged by x, it will displace a liquid of volume xa^2. Therefore, the buoyant force will be

$$F_\mathrm{b} = V\rho_\mathrm{w} g = a^2 x\rho_\mathrm{w} g\,.$$

The resulting force will then be (if we choose the direction against the direction of motion, i.e. upwards, as the positive direction)

$$F_\mathrm{t} = F_b - F_\mathrm{g} = a^2 x\rho_\mathrm{w} g - a^3\rho_\mathrm{k} g\,.$$

Hence, the resulting work can be obtained simply by integration

$$W = \int_0^h F_\mathrm{t}\,\mathrm{d}x = \int_0^h \left(a^2 x\rho_\mathrm{w} g - a^3\rho_\mathrm{k} g\right)\mathrm{d}x = \left[a^2\rho_\mathrm{w} g x^2/2 - a^3\rho_\mathrm{k} g x\right]_0^h = g a^2 h\left(\rho_\mathrm{w}\frac{h}{2} - \rho_\mathrm{k} a\right)\,.$$

Or, if we want to avoid integration, we can use the fact that the force varies linearly, so we can take the average force and multiply it by the total length of the trajectory. Its total length is h, the average force can be calculated, for instance, as the force in the middle of the motion (i.e. in $x = h/2$) as

$$\bar{F} = a^2\frac{h}{2}\rho_\mathrm{w} g - a^3\rho_\mathrm{k} g\,.$$

We could also calculate it as the average of the forces at the beginning and the end (this works precisely because the force varies linearly, otherwise this trick would not work). Then finally we get the expression for work as

$$W = \bar{F}s = \left(a^2 \frac{h}{2}\rho_\mathrm{w} g - a^3 \rho_\mathrm{k} g\right) h = g a^2 h \left(\rho_\mathrm{w} \frac{h}{2} - \rho_\mathrm{k} a\right) = 0.088\,\mathrm{J}\,.$$

It is also possible to use other tricks analogous to calculating the path of uniformly accelerated motion (such as calculating the area of a trapezoid). Everything gives us the same result.

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Problem H.2 ... air under water – very simplified 4 points

We have a cube of side length $a = 10\,\mathrm{cm}$ and density $\rho_\mathrm{k} = 150\,\mathrm{kg{\cdot}m^{-3}}$. Starting from a position where the bottom face of the cube is just touching the water surface, we submerge the cube in the water by pushing it straight down (without any rotation) by $h = 15\,\mathrm{cm}$. How much work is performed in this process? Assume that the surface area S of the water is enormous ($S \gg a^2$), so the water level does not change during the process. *...and as Lego was correcting...*

Using energy considerations

The cube has mass $m_\mathrm{k} = a^3\rho_\mathrm{k}$, so when it is submerged by $\Delta h_\mathrm{k} = -h$, its potential energy decreases by

$$\Delta E_\mathrm{pk} = m_\mathrm{k} g \Delta h_\mathrm{k} = -a^3 \rho_\mathrm{k} g h\,.$$

The whole cube ended up under the water surface, so we have displaced water with volume a^3, and mass $m_\mathrm{w} = a^3\rho_\mathrm{w}$. The centre of gravity of this block of water was $h - a/2$ below the surface before the immersion. Since we assume that the reservoir of water is large and thus the water level does not change when the cube is submerged, this water was displaced exactly to the level of the surface, so the height of its center of gravity increased by $\Delta h_\mathrm{w} = h - a/2$. Hence the potential energy of the water has increased by

$$\Delta E_\mathrm{pw} = m_\mathrm{w} g \Delta h_\mathrm{w} = a^3 \rho_\mathrm{w} g \left(h - \frac{a}{2}\right)\,.$$

So, in total, the potential energy increases by

$$\Delta E_\mathrm{p} = \Delta E_\mathrm{pw} - \Delta E_\mathrm{pk} = a^3 \rho_\mathrm{w} g \left(h - \frac{a}{2}\right) - a^3 \rho_\mathrm{k} g h = g a^3 \left(\rho_\mathrm{w}\left(h - \frac{a}{2}\right) - \rho_\mathrm{k} h\right) = 0.76\,\mathrm{J}\,.$$

Using forces

The cube is continuously affected by the gravitational force $F_\mathrm{g} = m_\mathrm{k} g = a^3 \rho_\mathrm{k} g$. At the same time, a buoyant force will be acting on it during the submersion. When the cube is submerged by $x < a$, it displaces a liquid of volume xa^2. Therefore, the buoyant force will be

$$F_\mathrm{b} = V \rho_\mathrm{w} g = a^2 x \rho_\mathrm{w} g\,.$$

Then the work necessary to get the cube under the surface can be obtained using the methods mentioned in the previous problem. By integration

$$W_1 = \int_0^a F_\mathrm{t}\,\mathrm{d}x = \int_0^a (a^2 x\rho_\mathrm{w} g - a^3\rho_\mathrm{k} g)\,\mathrm{d}x = [a^2\rho_\mathrm{w} g x^2/2 - a^3\rho_\mathrm{k} g x]_0^a = ga^4\left(\frac{1}{2}\rho_\mathrm{w} - \rho_\mathrm{k}\right) .$$

or, using the average force

$$W_1 = \bar{F}s = \left(a^2\frac{a}{2}\rho_\mathrm{w} g - a^3\rho_\mathrm{k} g\right)h = ga^4\left(\frac{1}{2}\rho_\mathrm{w} - \rho_\mathrm{k}\right) .$$

Consequently, when the whole cube is underwater, the volume of the displaced liquid is constant, so the total force is also constant $F_\mathrm{t} = a^3 g(\rho_\mathrm{w} - \rho_\mathrm{k})$. The cube has reached this state after moving a distance a, so with this constant force it has to move the remaining $h - a$, and thus it is still necessary to perform the work

$$W_2 = F_\mathrm{t}(h - a) = a^3 g(\rho_\mathrm{w} - \rho_\mathrm{k})(h - a) .$$

Finally, we obtain that the total work performed during the process is

$$W = W_1 + W_2 = ga^4\left(\frac{1}{2}\rho_\mathrm{w} - \rho_\mathrm{k}\right) + a^3 g(\rho_\mathrm{w} - \rho_\mathrm{k})(h - a) = ga^3\left(\rho_\mathrm{w}\left(h - \frac{a}{2}\right) - \rho_\mathrm{k} h\right) = 0.76\,\mathrm{J} .$$

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Problem H.3 ... air under water – less simplified — 5 points

We have a cube of side length $a = 10\,\mathrm{cm}$ and density $\rho_\mathrm{k} = 150\,\mathrm{kg{\cdot}m^{-3}}$. Starting from a position where the bottom face of the cube is just touching the water surface, we submerge the cube in a water container by pushing it straight down (without any rotation) by $h = 15\,\mathrm{cm}$. How much work is performed in this process? The cross-section of the container (i.e. the surface area) is $S = 300\,\mathrm{cm}^2$. *...Lego thought of different simplified versions of the problem...*

We will only discuss the solution using energy, as it is mathematically simpler. The cube has mass $m_\mathrm{k} = a^3\rho_\mathrm{k}$, so that its potential energy decreases by $\Delta h_\mathrm{k} = -h$ when it is submerged by $\Delta h_\mathrm{k} = -h$.

$$\Delta E_\mathrm{pk} = m_\mathrm{k} g\Delta h_\mathrm{k} = -a^3\rho_\mathrm{k} g h .$$

The whole cube ended up under water, so the displaced water is of volume a^3, and mass $m_\mathrm{w} = a^3\rho_\mathrm{w}$. The centre of gravity of this block was $h - a/2$ below the surface before the being submerged. The displaced water is now above the original surface level. Thus, the level has risen by the ratio of the volume of displaced water to the surface area $\Delta h_\mathrm{h} = a^3/S$, with the center of gravity of the displaced water being in the middle of this block. The displaced water has thus risen by its path to the surface and by a path that is above the original surface $\Delta h_\mathrm{w} = h - a/2 + \Delta h_\mathrm{h}/2 = h - a/2 + a^3/(2S)$. Thus the potential energy of water increased by

$$\Delta E_\mathrm{pw} = m_\mathrm{w} g\Delta h_\mathrm{w} = a^3\rho_\mathrm{w} g\left(h - \frac{a}{2} + \frac{a^3}{2S}\right) .$$

In total, the potential energy increases by

$$\Delta E_\mathrm{p} = \Delta E_\mathrm{pw} - \Delta E_\mathrm{pk} = a^3\rho_\mathrm{w} g \left(h - \frac{a}{2} + \frac{a^3}{2S}\right) - a^3\rho_\mathrm{k} gh$$
$$= ga^3 \left(\rho_\mathrm{w}\left(h - \frac{a}{2} + \frac{a^3}{2S}\right) - \rho_\mathrm{k} h\right) = 0.92\,\mathrm{J}\,.$$

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Problem H.4 ... air under water – still simplified 5 points

We have a cube of side length $a = 10\,\mathrm{cm}$ and density $\rho_\mathrm{k} = 150\,\mathrm{kg{\cdot}m^{-3}}$. Starting from a position where the bottom face of the cube is just touching the water surface, we submerge the cube in a water container by pushing it straight down (without any rotation) by $h = 6.0\,\mathrm{cm}$. How much work is performed in this process? The cross-section of the container (i.e. the surface area) is $S = 300\,\mathrm{cm}^2$. *...until Lego finished correcting.*

A solution using forces is possible, but more complex (although in the case of this problem the difference is not that significant).

The cube has mass $m_\mathrm{k} = a^3\rho_\mathrm{k}$, so upon submerging by $\Delta h_\mathrm{k} = -h$, its potential energy decreases by

$$\Delta E_\mathrm{pk} = m_\mathrm{k} g\Delta h_\mathrm{k} = -a^3\rho_\mathrm{k} gh\,.$$

We have displaced[5] water of volume a^2h and of mass $m_\mathrm{w} = a^2h\rho_\mathrm{w}$. The centre of mass of this block of water was $h/2$ below the surface before submerging. However, the displaced water must rise above the original surface and fit somewhere between the cube's walls and the walls of the container, forming a prism with base area $S - a^2$ and volume a^2h. The height of this prism will therefore be $\Delta h_\mathrm{h} = a^2h/(S - a^2) = 3\,\mathrm{cm}$,[6] with the centre of gravity of the displaced water being in the middle of this height.The change in the height of the displaced water's center of mass is thus given by the sum of the distance to the original surface ($h/2$) and the distance above it ($a^2h/2(S - a^2)$). Altogether, the total change in the potential energy of water will be

$$\Delta E_\mathrm{pw} = m_\mathrm{w} g\Delta h_\mathrm{w} = a^2h\rho_\mathrm{w} g\left(\frac{h}{2} + \frac{a^2h}{2(S - a^2)}\right) = \frac{1}{2}a^2h^2\rho_\mathrm{w} g\frac{S}{S - a^2}\,,$$

we can note that for $S \gg a^2$ the last fraction would approach 1, so we would get the same result as in the first problem of this hurry-up series.

In total, the potential energy increases by

$$W = \Delta E_\mathrm{pw} + \Delta E_\mathrm{pk} = \frac{1}{2}a^2h^2\rho_\mathrm{w} g\frac{S}{S - a^2} - a^3\rho_\mathrm{k} gh = ga^2h\left(\rho_\mathrm{w}\frac{h}{2}\frac{S}{S - a^2} - \rho_\mathrm{k} a\right) = 0.18\,\mathrm{J}\,.$$

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[5]Note, in this solution, by "displaced water", we only mean the portion of the cube's volume that is in the region where water was present before submersion. Thus, the portion of the volume that is now below the surface but above the original water level is not considered, as there was no water there, and we do not need to account for its displacement.

[6]It is important to check here that indeed $h + \Delta h_\mathrm{h} < a$, i.e., the cube will not be fully submerged.

Problem M.1 ... first ride through Blanka 3 points

A road pirate drives a car at $v_1 = 80.0\,\mathrm{km{\cdot}h^{-1}}$. After travelling one third of the distance over which the section measurement is taking place, he realises that he should maintain the maximum permissible speed $\bar{v} = 70.0\,\mathrm{km{\cdot}h^{-1}}$, which is measured as the average speed of travelling through the measured section. What is the highest speed v_2 he can travel for the next two-thirds of the track to maintain the maximum permissible speed? Neglect the time required to decelerate to the desired speed. Give the answer in $\mathrm{km{\cdot}h^{-1}}$. *Karel was thinking about speed.*

Denoting the total distance traveled as s, we can express the total time, and therefore the maximum speed that the road pirate could travel as follows

$$t = \frac{s}{\bar{v}} = \frac{1}{3}\frac{s}{v_1} + \frac{2}{3}\frac{s}{v_2}\,,$$

where the total distance cancels out. This simplifies the equation. We want to determine the velocity v_2, so we rearrange it to isolate v_2 on one side

$$\frac{2}{3}\frac{1}{v_2} = \frac{1}{\bar{v}} - \frac{1}{3}\frac{1}{v_1}\,,$$

which leads to

$$\frac{3}{2}v_2 = \frac{3\bar{v}v_1}{3v_1 - \bar{v}}\,.$$

From there, we find

$$v_2 = \frac{2\bar{v}v_1}{3v_1 - \bar{v}} \doteq 65.9\,\mathrm{km{\cdot}h^{-1}}\,.$$

If the pirate travels the first third of the distance at $80.0\,\mathrm{km{\cdot}h^{-1}}$, then he must drive the rest of the way at $65.9\,\mathrm{km{\cdot}h^{-1}}$ to maintain an average speed of $70.0\,\mathrm{km{\cdot}h^{-1}}$. Thus, this result is not simply $65.0\,\mathrm{km{\cdot}h^{-1}}$ as one might intuitively expect.

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Problem M.2 ... riding through Blanka... maybe... 4 points

Imagine that you need to decide how to travel to your cottage. You are choosing between a route of length $s_1 = 13\,\mathrm{km}$ traveled in $t_1 = 25\,\mathrm{min}$ and a route of length $s_2 = 20\,\mathrm{km}$ traveled in $t_2 = 20\,\mathrm{min}$. Assume that the gasoline consumption is constant $c = 6.7\,\mathrm{l}/(100\,\mathrm{km})$ and the price of gasoline is $p = 40.2\,\mathrm{CZK{\cdot}l^{-1}}$. It is said that time is money. What is the minimum value your time must have to make it more profitable for you to take the longer route? Give the result in Czech crowns per hour. *Karel was thinking about finances and the value of time.*

The real problem would of course be more complex. The traffic situation is dynamically changing, and even if the navigation advises us to take a faster route, an accident may occur on that section and we may be delayed. Conversely, a traffic jam may dissolve on another route that we did not choose. Our problem is a simple model with constant parameters. In a real situation, fuel consumption also varies depending on speed, and in urban areas, it fluctuates based on acceleration and deceleration at intersections.

Now, let us solve our specific problem. The cost of fuel consumed while traversing a route is determined by the product of the distance, fuel consumption, and fuel price. For the first, shorter route, the cost is

$$P_1 = s_1 c p \doteq 35.0\,\mathrm{CZK}\,.$$

For the second route, it is similarly

$$P_2 = s_2 c p \doteq 53.9\,\mathrm{CZK}\,.$$

To justify taking the longer route, the value of time, in units of CZK·h^{-1}, must exceed the difference in costs divided by the time taken, i.e.,

$$X > \frac{P_2 - P_1}{t_1 - t_2} = \frac{s_2 - s_1}{t_1 - t_2} c p \doteq 226\,\mathrm{CZK{\cdot}h^{-1}}\,.$$

To take the longer route in less time in this particular case, we have to value our time at more than 226 crowns per hour. In a real-world situation, the vehicle achieves its lowest fuel consumption when driving smoothly without breaking, at a speed that typically depends on the car's aerodynamics. This speed is usually claimed to be around $80\,\mathrm{km{\cdot}h^{-1}}$ to $90\,\mathrm{km{\cdot}h^{-1}}$ for normal passenger cars. Therefore, the difference in the fuel consumed on our two routes is likely to be smaller, so a lower estimate of our time would suffice.

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Problem M.3 ... changing speeds in Blanka 4 points

Imagine you are driving on a straight road with no restrictions due to traffic, but you want to adhere to the speed limits. On the road, the speed limits regularly alternates between $v_{50} = 50\,\mathrm{km{\cdot}h^{-1}}$ and $v_{70} = 70\,\mathrm{km{\cdot}h^{-1}}$ and the signs are always $d = 500\,\mathrm{m}$ apart. You drive in such way, that at the sign 70, you start to accelerate with acceleration $a = 1.2\,\mathrm{m{\cdot}s^{-2}}$ from the speed v_{50} to v_{70}. Conversely, you start decelerating with an acceleration of the same magnitude but in the opposite direction to reach a speed of v_{50} just at the 50 sign. Otherwise, you always drive at the maximum speed allowed in the section. What average speed in kilometers per hour will you reach? Ignore the length of the car. *Karel thought about averages.*

The described road ride consists of four repeating parts: a steady ride over the whole distance at v_{50}, acceleration to v_{70}, a steady ride at v_{70}, and deceleration back to v_{50}. The average speed will be determined from the total time T it takes for the car to travel the distance $2d$.

The time for the first part is easily calculated as

$$t_1 = \frac{d}{v_{50}} = 36\,\mathrm{s}\,.$$

From symmetry, the second and fourth parts will take the same amount of time, and we determine the time based on the acceleration from v_{50} to v_{70} as

$$t_2 = t_4 = \frac{v_{70} - v_{50}}{a} \doteq 4.63\,\mathrm{s}\,.$$

We should verify, whether we even have enough time to accelerate and decelerate, but we will do that now by finding out how much distance we still have to travel to determine the third time. The distance the car will travel during the time t_2 (and therefore also during t_4) is given by

$$\Delta s = \frac{1}{2}at_2^2 + v_{50}t_2 = \frac{1}{2}\frac{(v_{70}-v_{50})^2}{a} + v_{50}\frac{v_{70}-v_{50}}{a} = \frac{1}{2}\frac{v_{70}^2-v_{50}^2}{a} \doteq 77.2\,\mathrm{m}\,.$$

Then, in the third phase of motion, it is sufficient to travel the distance $d-2\Delta s$ at speed v_{70}, i.e.

$$t_3 = \frac{d-2\Delta s}{v_{70}} = \frac{d-\dfrac{v_{70}^2-v_{50}^2}{a}}{v_{70}} = \frac{d}{v_{70}} - \frac{v_{70}}{a} + \frac{v_{50}^2}{av_{70}} \doteq 17.8\,\mathrm{s}\,.$$

We obtain the total cycle time as a rather convoluted expression

$$T = t_1+t_2+t_3+t_4 = \frac{d}{v_{50}} + 2\frac{v_{70}-v_{50}}{a} + \frac{d}{v_{70}} - \frac{v_{70}}{a} + \frac{v_{50}^2}{av_{70}} =$$
$$= d\left(\frac{1}{v_{50}}+\frac{1}{v_{70}}\right) + \frac{1}{a}\left(v_{70}-2v_{50}+\frac{v_{50}^2}{v_{70}}\right) \doteq 63.0\,\mathrm{s}\,.$$

The average velocity is then given as

$$\bar{v} = \frac{2d}{T} = \frac{2}{\frac{1}{v_{50}}+\frac{1}{v_{70}}+\frac{1}{da}\left(v_{70}-2v_{50}+\frac{v_{50}^2}{v_{70}}\right)} \doteq 15.9\,\mathrm{m{\cdot}s^{-1}} = 57.1\,\mathrm{km{\cdot}h^{-1}}\,.$$

Maintaining this driving speed results in an average speed of $57.1\,\mathrm{km{\cdot}h^{-1}}$.

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Problem M.4 ... riding through Blanka... pirate style — 5 points

Imagine a road pirate, with a car of mass $m = 1\,290\,\mathrm{kg}$ and of engine power $P = 92.0\,\mathrm{kW}$, which drives at speed $v_0 = 70.0\,\mathrm{km{\cdot}h^{-1}}$ into the Blanka tunnel complex and decides to accelerate with maximum engine power. What speed would he achieve if he could accelerate this way for the entire length of the complex $d = 5\,502\,\mathrm{m}$? For simplicity, neglect the tunnel height changes and the drag forces (which in reality play an important role) and assume that the car has no limit on its maximum speed. Give the result in kilometers per hour.

Karel thought about car acceleration.

First of all, we must not forget to convert the velocity $v_0 \doteq 19.4\,\mathrm{m{\cdot}s^{-1}}$ into basic units. Since we can neglect drag forces and changes in height, we can start from the law of conservation of mechanical energy

$$E_k = Pt = \frac{1}{2}mv^2 \quad\Rightarrow\quad v = \sqrt{\frac{2Pt}{m}}\,,$$

where v is the velocity at time t. We can use this relation for velocity to integrate and obtain the position dependence on time. The bounds for our integration will be considered as the time when the car has a velocity v, which occurs at $t_0 = mv_0^2/(2P) \doteq 2.65\,\mathrm{s}$, to some time t_1

when we reach the end of the tunnel complex at a distance d from the start of the motion measurement. Thus, we get

$$d = \int_{t_0}^{t_1} \sqrt{\frac{2Pt}{m}}\,\mathrm{d}t = \frac{2}{3}\sqrt{\frac{2P}{m}}\left(\sqrt{t_1^3} - \sqrt{t_0^3}\right) = \frac{2}{3}\sqrt{\frac{2P}{m}}\left(\sqrt{t_1^3} - \sqrt{\left(\frac{mv_0^2}{2P}\right)^3}\right)\,.$$

We are now looking for the time t when the pirate exits the tunnel complex, so we can express t_1 from the previous equation as follows

$$\sqrt{t_1^3} = \left(\frac{mv_0^2}{2P}\right)^{3/2} + \frac{3}{2}d\sqrt{\frac{m}{2P}}\,,$$

$$t_1 = \left(\left(\frac{mv_0^2}{2P}\right)^{3/2} + \frac{3}{2}d\sqrt{\frac{m}{2P}}\right)^{2/3} \doteq 78.5\,\mathrm{s}\,.$$

The pirate reaches the end of the tunnel in approximately $78.5\,\mathrm{s}$, and his speed can be determined by substituting this value into the original relation for speed over time

$$v_1 = \sqrt{\frac{2Pt_1}{m}} = \sqrt{\frac{2P}{m}}\left(\left(\frac{mv_0^2}{2P}\right)^{3/2} + \frac{3}{2}d\sqrt{\frac{m}{2P}}\right)^{1/3}$$

$$= \sqrt[3]{v_0^3 + \frac{3P}{m}d} \doteq 106\,\mathrm{m{\cdot}s^{-1}} \doteq 381\,\mathrm{km{\cdot}h^{-1}}\,.$$

Thus, if the pirate kept his foot on the gas, and if there were no drag forces on the car, the road height remained constant, and the car could accelerate to that speed, he would reach a speed of $381\,\mathrm{km{\cdot}h^{-1}}$.

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